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VARIATIONAL APPROACH FOR COMPLEX MONGE-AMPÈRE EQUATIONS AND GEOMETRIC APPLICATIONS
[after Berman, Berndtsson, Boucksom, Eyssidieux, Guedj, Jonsson, Zeriahi, ...]

by **Jean-Pierre DEMAILLY**

INTRODUCTION

Monge-Ampère equations on compact Kähler manifolds can be solved by a variational method that is independent of Yau's theorem. The technique of [**16**] is based on the study of certain functionals (Ding-Tian, Mabuchi) on the space of Kähler metrics, and their geodesic convexity due to [**34**] and Berman-Berndtsson [**9**] in its full generality. Recent applications include the existence and uniqueness of Kähler-Einstein metrics on $\mathbb{Q}$-Fano varieties with log terminal singularities, given in [**15**], and a new proof by [**17**] of a uniform version of the Yau-Tian-Donaldson conjecture [**81**]. This provides a simpler route to the existence theorem for Kähler-Einstein metrics due to Chen-Donaldson-Sun [**36**], albeit with a stronger hypothesis. Our goal is to present the main ideas involved in this approach (starting from the basics!)

0.A. *Kähler metrics.* — A *Kähler manifold* (X,ω) is a complex manifold X of dimension $n=\dim_{\mathbb{C}} X$ endowed with a d-closed smooth positive $(1,1)$-form ω. In local holomorphic coordinates $(z_1,\ldots,z_n)$, one can write $\omega = i\sum_{1\le j,k\le n}\omega_{jk}(z)\,dz_j\wedge d\overline{z}_k$, i.e., $(\omega_{jk}(z))$ is a positive definite hermitian matrix at every point, and $d\omega=0$, so that ω is also a (real) symplectic structure on X. The holomorphic tangent bundle T_X is then equipped with the associated hermitian structure $h_\omega=\sum_{1\le j,k\le n}\omega_{jk}(z)\,dz_j\otimes d\overline{z}_k$. There is a unique connection ∇_h on T_X, called the Chern connection, such that h is ∇_h-parallel and $\nabla_h^{0,1}$ coincides with the $\overline{\partial}$ operator given by the complex structure. The Chern curvature tensor, which coincides with the Riemann curvature tensor in the Kähler case, is the $(1,1)$-form form with values in the bundle of endomorphisms of T_X, i.e., a section in $C^\infty(X,\Lambda^{1,1}T_X^*\otimes \mathrm{End}(T_X))$, given by

$$\Theta_{T_X,\omega} := \frac{i}{2\pi}\nabla_h^2 = i\sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}dz_j\wedge d\overline{z}_k\otimes \frac{\partial}{\partial z_\lambda}^{*}\otimes\frac{\partial}{\partial z_\mu}. \tag{0.1}$$

Its trace $\mathrm{Tr}(\Theta_{T_X,\omega}) = i\sum_{j,k,\lambda} c_{jk\lambda\lambda} dz_j \wedge d\overline{z}_k$ is also the curvature form of the anti-canonical line bundle $\Lambda^n T_X$ $(= -K_X$ in additive notation), and is by definition the *Ricci curvature* $\mathrm{Ricci}(\omega)$. A standard calculation gives

(0.2)

$$\mathrm{Ricci}(\omega) = \Theta_{\Lambda^n T_X,\Lambda^n\omega} = -dd^c \log\det(\omega_{jk}) \quad \text{where } d^c = \tfrac{1}{4i\pi}(\partial - \overline{\partial}),\ dd^c = \tfrac{i}{2\pi}\,\partial\overline{\partial}.$$

By definition, $\mathrm{Ricci}(\omega)$ is a closed real $(1,1)$-form, and its De Rham cohomology class is induced by the first Chern class $c_1(X) := c_1(T_X) = -c_1(K_X) \in H^2(X,\mathbb{Z})$.

0.B. *Kähler-Einstein metrics and the conjecture of Yau-Tian-Donaldson.* — A Kähler metric ω is said to be *Kähler-Einstein* if

$$\mathrm{Ricci}(\omega) = \lambda\omega \quad \text{for some } \lambda \in \mathbb{R}. \tag{0.3}$$

This requires $\lambda\omega \in c_1(X)$, hence (0.3) can be solved only when $c_1(X)$ is positive definite, negative definite or zero, and after rescaling ω by a constant, one can always assume that $\lambda \in \{0, 1, -1\}$. Let us fix some reference Kähler metric ω_0. Under the cohomological assumption $c_1(X) = \lambda\{\omega_0\} \in H^2(X,\mathbb{R})$, the $\partial\overline{\partial}$-lemma says that there is a function $f \in C^\infty(X,\mathbb{R})$ such that

$$\mathrm{Ricci}(\omega_0) - \lambda\omega_0 = dd^c f. \tag{0.4}$$

The potential f is defined modulo an additive constant, and we will normalize f so that $\int_X e^f\omega_0^n = \int_X \omega_0^n$. If we look for a solution $\omega = \omega_0 + dd^c\varphi$ of (0.3) in the same cohomology class as ω_0, Formula (0.2) yields $\mathrm{Ricci}(\omega) - \mathrm{Ricci}(\omega_0) = -dd^c \log(\omega_0 + dd^c\varphi)^n/\omega_0^n$, and the Kähler-Einstein condition (0.3) is reduced to solving the Monge-Ampère equation

$$(\omega_0 + dd^c\varphi)^n = e^{-\lambda\varphi + f}\omega_0^n. \tag{0.5}$$

• When $\lambda = -1$ and $c_1(X) < 0$, i.e., $c_1(K_X) > 0$, Aubin [**2**] has shown that there is always a unique solution, hence a unique Kähler metric $\omega \in c_1(K_X)$ such that

$$\mathrm{Ricci}(\omega) = -\omega.$$

This is a very natural generalization of the existence of constant curvature metrics on complex algebraic curves, implied by Poincaré's uniformization theorem in dimension 1.

• When $\lambda = 0$ and $c_1(X) = 0$, the celebrated result of [**85**] states that there exists a unique metric $\omega = \omega_0 + dd^c\varphi$ in the given cohomology class $\{\omega_0\}$ such that $\mathrm{Ricci}(\omega) = 0$ (solution of the Calabi conjecture [**28**], [**29**]). More generally, without any assumption on $c_1(X)$, [**85**] showed that the Monge-Ampère equation $(\omega_0 + dd^c\varphi)^n = e^f\omega_0^n$ has a unique solution whenever $\int_X e^f\omega_0^n = \int_X \omega_0^n$, in other words, one can prescribe the volume form $\omega^n = (\omega_0 + dd^c\varphi)^n$ to be any given volume form $e^f\omega_0^n > 0$ under the