415

ASTÉRISQUE

2020

SOME ASPECTS OF THE THEORY OF DYNAMICAL SYSTEMS: A TRIBUTE TO JEAN-CHRISTOPHE YOCCOZ

Volume I

On the minima of Markov and Lagrange Dynamical Spectra

Carlos Gustavo T. de A. Moreira

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Astérisque
415, 2020, p. 45–57
doi:10.24033/ast.1099

ON THE MINIMA OF MARKOV AND LAGRANGE DYNAMICAL SPECTRA

by

Carlos Gustavo T. de A. Moreira

Dedicated to Jean-Christophe Yoccoz

Abstract. — We consider typical Lagrange and Markov dynamical spectra associated to horseshoes on surfaces. We show that for a large set of real functions on the surface, the minima of the corresponding Lagrange and Markov dynamical spectra coincide, are isolated, and are given by the image of a periodic point of the dynamics by the real function. This solves a question by Jean-Christophe Yoccoz.

***Résumé* (Sur les minima des Spectres Dynamiques de Markov et Lagrange).** — Nous considérons des spectres dynamiques typiques de Lagrange et Markov associés aux fers à cheval des surfaces. Pour un grand ensemble de fonctions définies sur la surface à valeurs réelles, nous montrons que les minima des spectres dynamiques de Markov et Lagrange coïncident, sont isolés et sont l'image par la fonction d'un point périodique de la dynamique. Cela répond à une question de Jean-Christophe Yoccoz.

1. Introduction

The classical Lagrange spectrum (cf. [**2**]) is defined as follows: Given an irrational number α, according to Dirichlet's theorem the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$ has infinitely many rational solutions $\frac{p}{q}$. Markov and Hurwitz improved this result (cf. [**2**]), proving that, for all irrational α, the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$ has infinitely many rational solutions $\frac{p}{q}$.

2010 Mathematics Subject Classification. — 11J06, 37D20, 28A78.
Key words and phrases. — Markov and Lagrange dynamical spectra, regular Cantor sets, fractal dimensions.

On the other hand, for a fixed irrational α, better results can be expected. We associate, to each α, its best constant of approximation (Lagrange value of α), given by

$$\begin{aligned} k(\alpha) &= \sup\left\{k>0 : \left|\alpha-\frac{p}{q}\right| < \frac{1}{kq^2} \text{ has infinitely many rational solutions } \frac{p}{q}\right\} \\ &= \limsup_{\substack{|p|,q\to\infty \\ p\in\mathbb{Z},q\in\mathbb{N}}} |q(q\alpha-p)|^{-1} \in \mathbb{R}\cup\{+\infty\}. \end{aligned}$$

Then, we always have $k(\alpha)\geq\sqrt{5}$. The *Lagrange spectrum* is the set

$$L=\{k(\alpha):\alpha\in\mathbb{R}\setminus\mathbb{Q} \text{ and } k(\alpha)<\infty\}.$$

Let α be an irrational number expressed in continued fractions by $\alpha=[a_0,a_1,\dots]$. Define, for each $n\in\mathbb{N}$, $\alpha_n=[a_n,a_{n+1},\dots]$ and $\beta_n=[0,a_{n-1},a_{n-2},\dots]$. Using elementary continued fractions techniques it can be proved that

$$k(\alpha)=\limsup_{n\to\infty}(\alpha_n+\beta_n).$$

The study of the geometric structure of L is a classical subject, which began with Markov, proving in 1879 ([**3**]) that

$$L\cap(-\infty,3)=\left\{k_1=\sqrt{5}<k_2=2\sqrt{2}<k_3=\frac{\sqrt{221}}{5}<\cdots\right\}$$

where k_n is a sequence (of irrational numbers whose squares are rational) converging to 3 - more precisely, the elements k_n of $L\cap(-\infty,3)$ are the numbers the form $\sqrt{9-\frac{4}{z^2}}$, where z is a positive integer such that there are other positive integers x,y with $x\leq y\leq z$ and $x^2+y^2+z^2=3xyz$.

Another interesting set is the classical *Markov spectrum* defined by (cf. [**2**])

$$M=\left\{\inf_{(x,y)\in\mathbb{Z}^2\setminus(0,0)}|f(x,y)|^{-1} : f(x,y)=ax^2+bxy+cy^2 \text{ with } b^2-4ac=1\right\}.$$

It is possible to prove (cf. [**2**]) that L and M are closed subsets of the real line with $L\subset M$ and that $L\cap(-\infty,3)=M\cap(-\infty,3)$.

Both the Lagrange and Markov spectrum have a dynamical interpretation. This fact is an important motivation for our work.

Let $\Sigma=(\mathbb{N}^*)^{\mathbb{Z}}$ and $\sigma\colon\Sigma\to\Sigma$ the shift defined by $\sigma((a_n)_{n\in\mathbb{Z}})=(a_{n+1})_{n\in\mathbb{Z}}$. If $f\colon\Sigma\to\mathbb{R}$ is defined by $f((a_n)_{n\in\mathbb{Z}})=\alpha_0+\beta_0=[a_0,a_1,\dots]+[0,a_{-1},a_{-2},\dots]$, then

$$L=\left\{\limsup_{n\to\infty} f(\sigma^n(\underline{\theta})) : \underline{\theta}\in\Sigma\right\}$$

and

$$M=\left\{\sup_{n\in\mathbb{Z}} f(\sigma^n(\underline{\theta})) : \underline{\theta}\in\Sigma\right\}.$$

Notice that $\sqrt{5}$, which is the common minimum of L and M, is the image by f of the fixed point $(\dots,1,1,1,\dots)$ of the shift map σ.

This last interpretation, in terms of a shift, admits a natural generalization of Lagrange and Markov spectrum in the context of hyperbolic dynamics (at least in dimension 2, which is the focus of this work).

We will define, as in [**4**], the Markov and Lagrange dynamical spectra associated to a hyperbolic set as follows. Let M^2 be a surface and $\varphi\colon M^2 \to M^2$ be a diffeomorphism with $\Lambda \subset M^2$ a hyperbolic set for φ (which means that $\varphi(\Lambda) = \Lambda$ and there is a decomposition $T_\Lambda M^2 = E^s \oplus E^u$ such that $D\varphi|_{E^s}$ is uniformly contracting and $D\varphi\,|_{E^u}$ is uniformly expanding). In this paper we will asume that Λ is a *horseshoe*: it is a compact, locally maximal, hyperbolic invariant set of saddle type (and thus Λ is not an attractor nor a repellor, and is topologically a Cantor set). Let $f\colon M^2 \to \mathbb{R}$ be a continuous real function, then the *Lagrange Dynamical Spectrum* associated to (f,Λ) is defined by

$$L(f,\Lambda) = \left\{ \limsup_{n\to\infty} f(\varphi^n(x)) : x \in \Lambda \right\},$$

and the *Markov Dynamical Spectrum* associate to (f,Λ) is defined by

$$M(f,\Lambda) = \left\{ \sup_{n\in\mathbb{Z}} f(\varphi^n(x)) : x \in \Lambda \right\}.$$

Here we prove the following theorem, which solves a question posed by Jean-Christophe Yoccoz to the author in 1998:

Main Theorem. — *Let Λ be a horseshoe associated to a C^2-diffeomorphism φ. Then there is a dense set $H \subset C^\infty(M,\mathbb{R})$, which is is C^0-open, such that for all $f \in H$, we have*

$$\min L(f,\Lambda) = \min M(f,\Lambda) = f(p),$$

where $p = p(f) \in \Lambda$ is a periodic point of φ. Moreover, $f(p)$ is an isolated point both in $L(f,\Lambda)$ and in $M(f,\Lambda)$.

Remark. — In the previous statement, a horseshoe means a compact, locally maximal (which means that it is the maximal invariant set in some neighborhood of it), transitive hyperbolic invariant set of saddle type (and so it contains a dense subset of periodic orbits).

Acknowledgements. — I would like to thank Carlos Matheus, Davi Lima, Sergio Romaña and Sandoel Vieira for helpful discussions on the subject of this paper. I would also like to thank the anonymous referee for his very valuable comments and suggestions, which helped to substantially improve this work.

2. Preliminaries from dynamical systems

If Λ is a hyperbolic set associated to a C^2-diffeomorphism, then the stable and unstable foliations $\mathscr{F}^s(\Lambda)$ and $\mathscr{F}^u(\Lambda)$ are $C^{1+\varepsilon}$ for some $\varepsilon > 0$. Moreover, these foliations can be extended to $C^{1+\varepsilon}$ foliations defined on a full neighborhood of Λ(cf. the comments at the end of Section 4.1 of [**6**], pp. 60).

We will consider the following setting. Let Λ be a horseshoe of φ. Let us fix a geometrical Markov partition $\{R_a\}_{a\in\mathbb{A}}$ with sufficiently small diameter consisting of rectangles $R_a \simeq I_a^s \times I_a^u$ delimited by compact pieces I_a^s, resp. I_a^u, of stable, resp. unstable, manifolds of certain points of Λ (cf. [7, pp 129] or [6] for more details). The set $\mathbb{B} \subset \mathbb{A}^2$ of admissible transitions consist of pairs (a_0, a_1) such that $\varphi(R_{a_0}) \cap R_{a_1} \neq \emptyset$. So, we can define the following transition matrix B which induces the same transitions than $\mathbb{B} \subset \mathbb{A}^2$

$$b_{a_i a_j} = 1 \text{ if } \varphi(R_{a_i}) \cap R_{a_j} \neq \emptyset, \quad b_{a_i a_j} = 0 \quad \text{otherwise, for } (a_i, a_j) \in \mathbb{A}^2.$$

Let $\Sigma_{\mathbb{A}} = \{\underline{a} = (a_n)_{n\in\mathbb{Z}} : a_n \in \mathbb{A} \text{ for all } n \in \mathbb{Z}\}$. We can define the homeomorphism of $\Sigma_{\mathbb{A}}$, the shift, $\sigma : \Sigma_{\mathbb{A}} \to \Sigma_{\mathbb{A}}$ defined by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$.

Let $\Sigma_B = \left\{\underline{a} \in \Sigma_{\mathbb{A}} : b_{a_n a_{n+1}} = 1, \forall n \in \mathbb{Z}\right\}$, this set is a closed and σ-invariant subspace of $\Sigma_{\mathbb{A}}$. Still denote by σ the restriction of σ to Σ_B. The pair (Σ_B, σ) is called a subshift of finite type of $(\Sigma_{\mathbb{A}}, \sigma)$. Given $x, y \in \mathbb{A}$, we denote by $N_n(x, y, B)$ the number of admissible strings for B of length $n+1$, beginning at x and ending with y. Then the following holds

$$N_n(x, y, B) = b_{xy}^n.$$

In particular, since $\varphi|_{\Lambda}$ is transitive, there is $N_0 \in \mathbb{N}^*$ such that for all $x, y \in \mathbb{A}$, $N_{N_0}(x, y, B) > 0$.

Subshifts of finite type also have a sort of local product structure. First we define the local stable and unstable sets: (cf. [7, chap 10])

$$\begin{aligned} W^s_{1/3}(\underline{a}) &= \{\underline{b} \in \Sigma_B : \forall n \geq 0,\ d(\sigma^n(\underline{a}), \sigma^n(\underline{b})) \leq 1/3\} \\ &= \{\underline{b} \in \Sigma_B : \forall n \geq 0,\ a_n = b_n\}, \\ W^u_{1/3}(\underline{a}) &= \{\underline{b} \in \Sigma_B : \forall n \leq 0,\ d(\sigma^n(\underline{a}), \sigma^n(\underline{b})) \leq 1/3\} \\ &= \{\underline{b} \in \Sigma_B : \forall n \leq 0,\ a_n = b_n\}, \end{aligned}$$

where $d(\underline{a}, \underline{b}) = \sum_{n=-\infty}^{\infty} 2^{-(2|n|+1)} \delta_n(\underline{a}, \underline{b})$ and $\delta_n(\underline{a}, \underline{b})$ is 0 when $a_n = b_n$ and 1 otherwise.

So, if $\underline{a}, \underline{b} \in \Sigma_B$ and $d(\underline{a}, \underline{b}) < 1/2$, then $a_0 = b_0$ and $W^u_{1/3}(\underline{a}) \cap W^u_{1/3}(\underline{b})$ is a unique point, denoted by the bracket $[\underline{a}, \underline{b}] = (\ldots, b_{-n}, \ldots, b_{-1}, b_0, a_1, \ldots, a_n, \ldots)$.

If φ is a diffeomorphism of a surface (2-manifold), then the dynamics of φ on Λ is topologically conjugate to a subshift Σ_B defined by B, namely, there is a homeomorphism $\Pi \colon \Sigma_B \to \Lambda$ such that, the following diagram commutes

$$\begin{array}{ccc} \Sigma_B & \xrightarrow{\ \sigma\ } & \Sigma_B \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi} \\ \Lambda & \xrightarrow{\ \varphi\ } & \Lambda \end{array} \qquad \text{i.e.,} \quad \varphi \circ \Pi = \Pi \circ \sigma.$$

Moreover, Π is a morphism of the local product structure, that is, $\Pi[\underline{a}, \underline{b}] = [\Pi(\underline{a}), \Pi(\underline{b})]$, (cf. [7, chap 10]).

If $p = \Pi(\theta) \in \Lambda$, we say that θ is the *kneading sequence* of p.