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QUENCHED AND ANNEALED TEMPORAL LIMIT THEOREMS FOR CIRCLE ROTATIONS

by

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In memory of Jean-Christophe Yoccoz

Abstract. — Let $h(x) = \{x\} - \frac{1}{2}$. We study the distribution of $\sum_{k=0}^{n-1} h(x + k\alpha)$ when x is fixed, and n is sampled randomly uniformly in $\{1, \dots, N\}$, as $N \to \infty$. Beck proved in **[2, 3]** that if $x = 0$ and α is a quadratic irrational, then these distributions converge, after proper scaling, to the Gaussian distribution. We show that the set of α where a distributional scaling limit exists has Lebesgue measure zero, but that the following *annealed* limit theorem holds: Let (α, n) be chosen randomly uniformly in $\mathbb{R}/\mathbb{Z} \times \{1, \dots, N\}$, then the distribution of $\sum_{k=0}^{n-1} h(k\alpha)$ converges after proper scaling as $N \to \infty$ to the Cauchy distribution.

***Résumé* (Théorèmes limites temporels modifiés pour les rotations du cercle).** — Soit $h(x) = \{x\} - \frac{1}{2}$. On étudie la distribution de $\sum_{k=0}^{n-1} h(x + k\alpha)$ pour x fixé et n tiré au hasard uniformément dans $\{1, \dots, N\}$, quand $N \to \infty$. Beck a montré dans [2, 3] que pour $x = 0$ et α irrationnel quadratique, ces distributions convergent, après un changement d'échelle approprié, vers une distribution gaussienne. Nous montrons que l'ensemble des α pour lesquels la distribution limite après changement d'échelle existe est de mesure de Lebesgue nulle, mais qu'on a le théorème limite modifié suivant: soit (α, n) choisi au hasard uniformément dans $\mathbb{R}/\mathbb{Z} \times \{1, \dots, N\}$, alors la distribution de $\sum_{k=0}^{n-1} h(k\alpha)$ converge après un changement d'échelle approprié quand $N \to \infty$ vers la distribution de Cauchy.

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1. Introduction

We study the centered ergodic sums of functions $h : \mathbb{T} \to \mathbb{R}$ for the rotation by a an irrational angle α

$$S_n(\alpha, x) = \left(\sum_{k=1}^{n} h(x + k\alpha)\right) - n \int_{\mathbb{T}} h(z)dz. \tag{1.1}$$

Weyl's equidistribution theorem says that for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and for every h Riemann integrable, $\frac{1}{n} S_n(\alpha, x) \xrightarrow[n\to\infty]{} 0$ uniformly in x. We are interested in higher-order asymptotics. We aim at results which hold for a set of full Lebesgue measure of α.

If h is sufficiently smooth, then $S_n(\alpha, x)$ is bounded for almost every α and all x (see **[12]** or Appendix A). The situation for piecewise smooth h is more complicated, and not completely undertstood even for functions with a single singularity.

Setup. — Here we study (1.1), for the simplest example of a piecewise smooth function with one discontinuity on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$h(x) = \{x\} - \frac{1}{2}. \tag{1.2}$$

The *fractional part* $\{x\}$ is the unique $t \in [0, 1)$ s.t. $x \in t + \mathbb{Z}$.

Case (1.2) is sufficient for understanding the behavior for typical α for all functions $f(t)$ on $\mathbb{T}$ which are differentiable everywhere except one point x_0, and whose derivative on $\mathbb{T}\setminus\{x_0\}$ extends to a function of bounded variation on $\mathbb{T}$. This is because of the following result proven in Appendix A:

Proposition 1.1. — *If $f(t)$ is differentiable on $\mathbb{T} \setminus \{x_1, \ldots, x_\nu\}$ and f' extends to a function with bounded variation on $\mathbb{T}$, then there are $A_1, \ldots, A_\nu \in \mathbb{R}$ s.t. for a.e. α there is $\varphi_\alpha \in C(\mathbb{T})$ s.t. for all $x \neq x_i$,*

$$f(x) = \sum_{i=1}^{\nu} A_i h(x + x_i) + \int_{\mathbb{T}} f(t)dt + \varphi_\alpha(x) - \varphi_\alpha(x + \alpha).$$

Of course there are many functions h for which Proposition 1.1 holds. The choice (1.2) is convenient, because of its nice Fourier series.

Methodology. — $S_n(\alpha, x)$ is very oscillatory. Therefore, instead of looking for simple asymptotic formulas for $S_n(\alpha, x)$, which is hopeless, we will look for simple *scaling limits for the distribution* of $S_n(\alpha, x)$ when x, or α, or n (or some of their combinations) are randomized. There are several natural ways to carry out the randomization:

(1) *Spatial* vs *temporal* limit theorems: In a spatial limit theorem, the initial condition x chosen randomly from the space $\mathbb{T}$. In a temporal limit theorem, the initial condition x is fixed, and the "time" n is chosen randomly uniformly in $\{1, \ldots, N\}$ as $N \to \infty$. Neither limit theorem implies the other, see **[10]**.

(2) *Quenched* vs *annealed* limit theorems: In a quenched limit theorem, α is fixed. In an annealed limit theorem α is randomized. The terminology is motivated by the theory of random walks in random environment; the parameter α is the "environment parameter."

We indicate what is known and what is still open in our case.

Known results on spatial limit theorems:— The quenched spatial limit theorem fails; the annealed spatial limit theorem holds.

The failure of the *quenched* spatial limit theorem is very general. It follows from the Denjoy-Koksma inequality that there are no quenched spatial distributional limit theorems for any rotation by $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and every function of bounded variation which is not a coboundary (e.g., $h(x) = \{x\} - \frac{1}{2}$). In the coboundary case, the spatial limit theorem is trivial. Many people have looked for weaker quenched versions of spatial distributional limit theorem (e.g., along special subsequences of "times"). See **[7, 10]** for references and further discussion.

The *annealed* spatial limit theorem is a famous result of Kesten.

***Theorem 1.2* ([14]).** — *If (x, α) is uniformly distributed on $\mathbb{T} \times \mathbb{T}$ then the distribution of $\frac{S_n(\alpha,x)}{\ln n}$ converges as $n \to \infty$ to a symmetric Cauchy distribution: $\exists \boldsymbol{\rho}_1 \neq 0$ s.t. for all $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n(\alpha, x)}{\ln n} \leq t\right) = \frac{1}{2} + \frac{\arctan(t/\boldsymbol{\rho}_1)}{\pi}.$$

See **[14]** for the value of constant $\boldsymbol{\rho}_1$. The same result holds for $h(x) = 1_{[0,\beta)}(\{x\}) - \beta$ with $\beta \in \mathbb{R}$, with different $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_1(\beta)$ **[14, 15]**.

Known results on temporal limit theorems:— Quenched temporal limit theorems are known for special α; There were no results on the annealed temporal limit theorem until this work.

The first temporal limit theorem for an irrational rotation (indeed for any dynamical system) is due to J. Beck **[2, 3]**. Let

$$M_N(\alpha, x) := \frac{1}{N} \sum_{n=1}^{N} S_n(\alpha, x) \ , \ \mathscr{S}_n(\alpha, x) := S_n(\alpha, x) - M_N(\alpha, x).$$

***Theorem 1.3* (Beck).** — *Let α be an irrational root of a quadratic polynomial with integer coefficients. Fix $x = 0$. If n is uniformly distributed on $\{1 \dots N\}$ then $\frac{\mathscr{S}_n(\alpha,x)}{\sqrt{\ln N}}$ converges to a normal distribution as $N \to \infty$.*

A similar result holds for the same x and α with $h(x) = \{x\} - \frac{1}{2}$ replaced by $1_{[0,\beta)}(\{x\}) - \beta$, $\beta \in \mathbb{Q}$ **[2, 3]**. **[1, 10]** extended this to all $x \in [0, 1)$. A remarkable recent paper by Bromberg & Ulcigrai **[5]** gives a further extension to all x, *all irrational α of bounded type*, and for an uncountable collection of β (which depends on α). Recall that the set of α of bounded type is a set of full Hausdorff dimension **[13]**, but zero Lebesgue measure **[16]**.

This paper. — We show that for $h(x) = \{x\} - \frac{1}{2}$, the quenched temporal limit theorem fails for a.e. α, but that the annealed temporal limit theorem holds. See §2 for precise statements.

Heuristic overview of the proof. — When we expand the ergodic sums of h into Fourier series, we find that the resulting trigonometric series can be split into the contribution of "resonant" and "non-resonant" harmonics.

The non-resonant harmonics are many in number, but small in size. They tend to cancel out, and their total contribution is of order $\sqrt{\ln N}$. It is natural to expect that this contribution has Gaussian statistics. If α has bounded type, all harmonics are non-resonant, and as Bromberg and Ulcigrai show in the case $1_{[0,\beta)} - \beta$ the limiting distribution is indeed Gaussian.

The resonant harmonics are small in number, but much larger in size: individual resonant harmonics have contribution of order $\ln N$. For typical α, the number, strength, and location of the resonant harmonics changes erratically with N in a non-universal way. This leads to the failure of temporal distributional limit theorems for typical α.

We remark that a similar obstruction to quenched limit theorems have been observed before in the theory of random walks in random environment [**9, 18, 6**].

To justify this heuristic we fix N and compute the distribution of resonances when α is uniformly distributed. Since the distribution of resonances is non-trivial, changing a scale typically leads to a different temporal distribution proving that there is no limit as $N \to \infty$. As a by-product of our analysis we obtain some insight on the frequency with which a given limit distribution occurs.

Functions with more than one discontinuity. — In a separate paper [**11**] we use a different method to show that given a piecewise smooth discontinuous function with arbitrary finite number of discontinuities, the quenched temporal limit theorems fails for Lebesgue almost all α. But this method does not provide an annealed result, and it does not give us as detailed information as we get here on the scaling limits which appear along subsequences for typical α.

2. Statement of results

Fix $x \in \mathbb{T}$ arbitrary. Let $S_n(\alpha, x) := \sum_{k=1}^{n} h(x + k\alpha)$, and

$$M_N(\alpha, x) = \frac{1}{N} \sum_{n=1}^{N} S_n(\alpha, x),$$

$$\mathscr{S}_n(\alpha, x) = S_n(\alpha, x) - M_N(\alpha, x).$$