416 ASTÉRISQUE

2020

SOME ASPECTS
OF THE THEORY OF DYNAMICAL SYSTEMS:
A TRIBUTE TO JEAN-CHRISTOPHE YOCCOZ

Volume II

On Roth type conditions, duality and central Birkhoff sums for i.e.m.

Stefano Marmi & Corinna Ulcigrai & Jean-Christophe Yoccoz

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Astérisque
416, 2020, p. 65–132
doi:10.24033/ast.1111

ON ROTH TYPE CONDITIONS, DUALITY AND CENTRAL BIRKHOFF SUMS FOR I.E.M.

by

Stefano Marmi, Corinna Ulcigrai & Jean-Christophe Yoccoz

Abstract. — We introduce two Diophantine conditions on rotation numbers of interval exchange maps (i.e.m.) and translation surfaces: the *absolute Roth type condition* is a weakening of the notion of Roth type i.e.m., while the *dual Roth type* condition is a condition on the *backward* rotation number of a translation surface. We show that results on the cohomological equation previously proved in **[38]** for restricted Roth type i.e.m. (on the solvability under finitely many obstructions and the regularity of the solutions) can be extended to restricted *absolute* Roth type i.e.m. Under the dual Roth type condition, we associate to a class of functions with *subpolynomial* deviations of ergodic averages (corresponding to relative homology classes) *distributional* limit shapes, which are constructed in a similar way to the *limit shapes* of Birkhoff sums associated in **[36]** to functions which correspond to positive Lyapunov exponents.

2010 Mathematics Subject Classification. — 37E05; 11K60, 37A20, 37E35.
Key words and phrases. — Diophantine conditions, rotation number, Roth type, interval exchange maps, Birkhoff sums, deviation of ergodic averages.

In August 2010 J.C.Y. wrote a first text (12 pages) containing the results obtained on dual Birkhoff sums with the title "On Birkhoff sums for i.e.m." (a further version of the same text was written in March 2011). This constitutes the heart of Section 4 of this paper. Notable progress was made during a visit of S.M. and J.C.Y. to C.U. in Bristol in October 2014, when the homological interpretation emerged and J.C.Y. wrote a new version of this draft introducing the notion of KZ-hyperbolic translation surfaces (this homological interpretation is included in Section 6). Motivated by the discussions in Bristol, the notion of absolute Roth type i.e.m. emerged and was developed during further meetings in of S.M. with J.C.Y. in Paris in December 2014, March 2015 and September 2015. This notion, together with the completeness of backward rotation numbers, is the object of the text "Absolute Roth Type and Backward Rotation Number" (14 pages) written by J.C.Y. in April 2015. This text formed the basis for Section 3 and parts of it are here included as two Appendixes (Appendix A and Appendix B). The final version of this manuscript was prepared by S.M. and C.U. who are fully and solely responsible for any mistake or imprecision.
Jean-Christophe discussed publicly the results obtained in our collaboration in his talk "Problèmes de petits diviseurs pour les échanges d'intervalles" on May 20, 2015 given at the "Journée Surfaces plates" held at the Institut Galilée of the University of Paris 13 (as Carlos Matheus informed us) and also during his talk "Diophantine conditions for interval exchange map" on September 28, 2015 given in Oxford as part of the Worshop *Geometry and Dynamics on Moduli Spaces* at the 2015 Clay Research conference.

Résumé **(Sur les conditions de type Roth, la dualité et les sommes de Birkhoff centrées pour les échange d'intervalles)**

Nous introduisons deux conditions diophantiennes pour les nombres de rotation des transformations d'échange d'intervalles (i.e.m.) et des surfaces de translation: la condition absolue de type Roth est un affaiblissement de la notion i.e.m de type Roth, tandis que la condition duale de type Roth est une condition sur le nombre de rotation en arrière d'une surface de translation.

Nous montrons que les résultats sur l'équation cohomologique prouvés précédemment dans [38] pour les i.e.m. de type Roth restreint (sur la solvabilité en supposant un nombre fini d'obstructions et la régularité des solutions) peuvent être étendues aux i.e.m. de type Roth absolu restreint. Sous la condition duale de type Roth, nous associons des formes limites (limit shapes) distributionnelles à une classe de fonctions avec des déviations sous-polynomiales des moyennes ergodiques (correspondantes aux classes d'homologie relatives), qui sont construites de manière similaire aux formes limites des sommes de Birkhoff associées dans [36] aux fonctions qui correspondent aux exposants de Lyapunov positifs.

1. Introduction

Diophantine conditions play a central role in the study of the dynamics of rotations of the circle, diffeomorphisms of the circle and more in general area-preserving flows on tori. These conditions, which convey information on how well a rotation number α can be approximated by rationals (and hence on small divisors problems), are often expressed in terms of growth rates for the entries of the continued fraction expansion of α.

In the study of dynamics on surfaces, one often requires similar Diophantine conditions on interval exchange maps and linear flows. Passing from genus one to higher genus, a natural generalization of linear flows on tori is indeed provided by linear flows on *translation surfaces* (see § 2.2 for definitions); *interval exchange maps*, which will be shortened throughout this paper by *i.e.m.*, are piecewise isometries which, analogously to rotations in genus one, arise are Poincaré maps of linear flows (see § 2.1 for the definition). An algorithm which plays in this context an analogous role to the continued fraction expansion is the *Rauzy-Veech induction*, first introduced **[40, 45]** and used since then as an essential tool for proving many results on the ergodic and spectral properties of i.e.m., flows on surfaces and rational billiards, see for example **[46, 53, 35, 25, 2, 1, 43, 44, 37, 12, 11, 41, 29]**.

Diophantine conditions on i.e.m. can be prescribed imposing conditions on the behaviour of the Rauzy-Veech induction matrices and the related (extended) Kontsevich-Zorich cocycle. In this spirit, in their work **[35]** on the cohomological equation for i.e.m., S. M. , P. Moussa and J.-C. Y., define a Diophantine condition on i.e.m. which generalize the notion of Roth type rotation and under which they show that the cohomological equation can be solved under finitely many obstructions (see Theorem 3.14 for a generalized statement). After Forni's celebrated paper **[22]** on the cohomological equation associated to linear flows on surfaces of higher genus, this was the first result giving an explicit diophantine condition sufficient to guarantee the existence of a solution to the cohomological equation. The i.e.m. which satify this condition are called *Roth type* i.e.m. (or i.e.m. of Roth type) and have full measure (as shown in [**35**], see also [**34**] for the sketch of a simpler proof based on the results in [**3**]). A reformulation and a strenghthening of the Roth type condition (namely, *restricted* Roth type) were then defined in [**37**] also for *generalized* i.e.m. and provide the Diophantine condition under which a linearization result is proved in [**37**].

Two new Diophantine conditions related to the Roth type condition for i.e.m. are introduced in this paper, for the applications that we explain in § 1.1 and § 1.2 below. More precisely (in § 3) we introduce the notion of *absolute Roth type* i.e.m., thus defining a class of i.e.m. which include and generalize Roth type i.e.m. but for which the results mentioned before on linearization and the cohomological equation still hold. We then define the notion of *dual Roth type* for a translation surface (or more precisely, for its suspension data or backward rotation number, see § 4). This is a Diophantine condition which is *dual* to the Roth type condition for i.e.m. in a sense which will be made precise further on (see § 4 and § 6). Let us now explain the motivation for introducing these conditions and the results which we proved assuming them, starting with the notion of dual Roth type.

1.1. Dual Roth type and distributional limit shapes. — Results on deviations of ergodic averages and ergodic integrals are a central part of the study of i.e.m. and translation flows, see for example the works by Zorich [**53, 52**], Forni [**23**], Avila-Viana [**4**] and Bufetov [**11**] among others. Deviations of ergodic averages, i.e., the oscillations of the Birkhoff sum $S_n f(x) := \sum_{k=0}^{n-1} f(T^k x)$ of a function $f : [0,1] \to \mathbb{R}$ of zero average over the orbit of (typical) point $x \in [0,1]$ under a i.e.m. $T : [0,1] \to [0,1]$ are of polynomial nature. In [**53**] Zorich shows for example that for a typical i.e.m. T and any mean-zero function f constant on the intervals exchanged by T, we have $S_n f(x) = O(x^\nu)$ for some power exponent $\nu < 1$; more precisely for a full measure set of T there exists $\nu = \nu(f)$ such that for all $x \in [0,1]$,

$$\limsup_{n\to+\infty} \frac{\log S_n f(x)}{\log n} = \nu. \tag{1.1}$$

Remark also that if f is a *coboundary* for T with bounded transfer function, i.e., $f = g \circ T - T$ where the *transfer function* $g : [0,1] \to \mathbb{R}$ is bounded, then Birkhoff sums $S_n f(x)$ are uniformely bounded (and in particular $\nu = 0$). The power exponent can be understood in terms of Lyapunov exponents of (a suitable acceleration) of the Kontsevich-Zorich cocycle associated to the Rauzy-Veech induction [**54, 52**]. In particular, ν is a ratio of Lyapunov exponents and depends on the position of the piecewise constant function f (identified with a vector of $\mathbb{R}^d$) with respect to the Oseledets filtration of the Kontsevich-Zorich cocycle. Using this interpretation, it follows from the work of Forni [**23**] (see also [**4**]) that, for a typical choice of function f, ν is positive; furthermore one also has $\nu < 1$ as an immediate consequence of the work of Veech [**48**] (see also [**23**] for a more general result)). A powerful result of similar nature for ergodic integrals of smooth area-preserving flows was proved by Forni in [**23**]: the *power spectrum* of ergodic integrals is related to Lyapunov exponents of the Kontsevich-Zorich cocycle and Forni's invariant distributions.

A finer analysis of the behaviour of Birkhoff sums or integrals, beyond the *size* of oscillations, appears in the works [**11, 36**]. In [**36**], motivated by the study of wandering intervals in affine i.e.m., S. M., P. Moussa and J.C.Y. introduced an object called *limit shape* and used it to describe the *shape* of ergodic sums (see § 3.4 and § 3.7.3 in [**36**]). Roughly speaking these are obtained by looking at suitably rescaled Birkhoff sums, where time is renormalized according to the leading Laypunov exponent of the Kontsevich-Zorich cocycle, whereas the range of the sum is renormalized using one of the other positive exponents, according to the choice of f. After this double rescaling one obtains a sequence of shapes exponentially converging (in the Hausdorff metric) to the graph of a Hölder function. In [**11**], Bufetov studies limit theorems for ergodic integrals of translation flows and describe weak limit distributions in terms of objects that he calls *Hölder cocycles* (or, in the context of Markov compacta, *finitely-additive measures*) and turn out to be *dual* to Forni's invariant distributions (see [**11**] for details). We remark that limit shapes and Hölder cocycles, despite having been introduced independently, are intrinsically related: limit shapes are essentially *graphs* of Hölder cocycles along flow leaves. Let us remark that similar results can also be proved for horocycle flows on negatively curved surfaces, see [**10**] were the existence of Hölder cocycles is proved in this context.

Both limit shapes and Hölder cocycles are associated to functions which display *truly polynomial* deviations, i.e., for which the exponent ν in (1.1) is strictly positive. More precisely, from the work of Forni [**23**] and Avila-Viana [**4**] it follows that for a typical i.e.m. T with d exchanged subintervals, the extended Kontsevich-Zorich cocycle has g positive Lyapunov exponents, g negative and $s-1$ zero ones, where $d = 2g+s-1$ and g and s can be computed from the combinatorics of T (g is the genus and s is the number of marked points of any translation surface which suspends T, see § 2.2). For typical i.e.m. T, functions which are *coboundaries* with bounded transfer functions and hence have *bounded* Birkhoff sums, can be associated to the *stable* space of the Kontsevich-Zorich cocycle, which correspond to *negative* Lyapunov exponents.

We construct in this paper objects similar to the limit shapes introduced in [**36**] for functions which display *subpolynomial* deviations of ergodic averages, i.e., functions for which $\nu = \nu(f)$ in (1.1) is equal to zero, but are not coboundaries. An important example of this type of function arise when considering rotations of the circle (which correspond to i.e.m. with $d = 2$) and a mean zero function $\chi = \chi_{[0,\beta]} - \beta$, where $\chi_{[0,\beta]}$ is the characteristic function of the interval $[0, \beta]$. The function χ can be seen as a piecewise constant function on a i.e.m. with $d = 3$. It is well known that in this case $S_n\chi$ display logarithmic deviations (which can be described for example in terms of the Ostrowski expansion of β with respect to α, see for example [**7**]). Results on the subdiffusive behaviour of these Birkhoff sums were proved for example in [**27, 14, 15**] (see also [**8**] for a related result in the context of substitutions). Celebrated results