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*Topological modular forms*

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# TOPOLOGICAL MODULAR FORMS [after Hopkins, Miller, and Lurie]

by **Paul G. GOERSS**

## INTRODUCTION

In the early 1970s, Quillen [**55**] noticed a strong connection between 1-parameter formal Lie groups and cohomology theories with a natural theory of Chern classes. The algebraic geometry of these formal Lie groups allowed Morava, Ravenel, and others to make predictions about large scale phenomena in stable homotopy theory, and the resulting theorems completely changed the field. For example, the solution of Ravenel's nilpotence conjectures by Devinatz, Hopkins, and Smith ([**18**] and [**29**]) was one of the great advances of the 1980s.

An example of a 1-parameter formal Lie group can be obtained by taking the formal neighborhood of the identity in a smooth algebraic group of dimension one. The additive group and the multiplicative group correspond to ordinary cohomology and complex $K$-theory respectively, and the only other algebraic groups of dimension 1 are elliptic curves. This class is different because elliptic curves can come in families over a base scheme $S$ and the geometry of the fibers can vary significantly as we move through $S$. Thus there are many elliptic cohomology theories and it should be possible to produce them in families over schemes. In retrospect, the realization, by Hopkins and Miller, of a good theory of elliptic cohomology theories provided a centerpiece for the emerging field of derived algebraic geometry.

Derived algebraic geometry has origins in a number of diverse sources. In geometry, there is the work of Serre on multiplicities in intersection theory [**61**] and the work of Illusie [**33**] on the cotangent complex. For an overview of the roots in stable homotopy theory, the article [**27**] is very useful. In algebraic $K$-theory, again originating with Quillen, it was important very early to consider algebraic varieties and schemes with sheaves of generalized ring objects and, indeed, it was mathematicians in this field who first wrote down a systematic theory [**34**]. Thanks to work of Toën, Vezzosi [**66**], and Lurie [47], we now have a fairly mature theory. The purpose here is to concentrate on the example of elliptic cohomology theories; that is, I would like to make precise the following statement of a theorem of Mike Hopkins and Haynes Miller, as refined by Jacob Lurie: the compactified Deligne-Mumford moduli stack of elliptic curves is canonically and essentially uniquely an object in derived algebraic geometry. The homotopy global sections of this derived stack form the ring spectrum of topological modular forms.

## 1. AN OVERVIEW

### 1.1. The moduli stack of elliptic curves

In the late 1960s, Deligne and Mumford [15] defined a moduli object $\mathcal{M}_g$ for algebraic curves of genus $g$. Thus, morphisms $X \to \mathcal{M}_g$ from a scheme $X$ to $\mathcal{M}_g$ are in one-to-one correspondence with smooth proper morphisms

$$q : C \longrightarrow X$$

of relative dimension 1 such that each fiber is a curve of genus $g$. It was known that $\mathcal{M}_g$ could not be a scheme; one way to see this is to note that automorphisms of the fibers of $q$ do not vary nicely with the fiber. However, Deligne and Mumford noticed that $\mathcal{M}_g$ exists if we enlarge the category of schemes slightly to include what we now call Deligne-Mumford stacks. From this example, the whole theory of algebraic stacks emerged. There is an extremely brief exposition on algebraic stacks at the beginning of Section 3.1.

From this collection, we single out the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves. These are curves of genus 1 with 1 marked point; that is diagrams of the form

$$C \underset{e}{\overset{q}{\rightleftarrows}} X$$

where $q$ is a family of smooth curves of genus 1 over $X$ and $e$ is a section identifying a distinguished point in each fiber. A classical, but still remarkable, property of these curves is that $C$ becomes an abelian group over $X$: there is a canonical commutative group multiplication on $C$ with $e$ as the unit. Furthermore, any morphism of elliptic curves is a group homomorphism. There are a number of reasons for singling out this stack: elliptic curves are central to algebraic number theory, for example. For algebraic topologists, the formal neighborhood $C_e$ of $e$ in $C$ gives a family of 1-parameter formal Lie groups ("formal groups") which, in turn, gives rise to a rich family of elliptic cohomology theories.

The stack $\mathcal{M}_{1,1}$ is not compact – more precisely, it is not proper over $\mathbb{Z}$. Indeed, the morphism to the affine line

$$j : \mathcal{M}_{1,1} \longrightarrow \mathbb{A}^1$$

which assigns to each elliptic curve $C$ the $j$-invariant $j(C)$ is almost (but not quite) a covering map. Deligne and Mumford found a canonical compactification $\bar{\mathcal{M}}_{e\ell\ell}$ of $\mathcal{M}_{1,1}$ which now classifies generalized elliptic curves with, possibly, nodal singularities. In an extensive study of this stack [**16**] Deligne and Rapoport showed that the $j$-invariant extends to a morphism

$$j : \bar{\mathcal{M}}_{e\ell\ell} \longrightarrow \mathbb{P}^1.$$

## 1.2. Derived schemes

The basic objects of algebraic geometry are schemes, which are locally ringed spaces $(X, \mathcal{O}_X)$ which are locally isomorphic to the prime spectrum of some ring. Thus, among other things, $X$ is a space and $\mathcal{O}_X$ is a sheaf of rings on $X$. The basic idea of derived algebraic geometry is to replace rings by some more generalized ring object. For example, Serre [**61**] considered schemes with a sheaf of commutative differential graded algebras. This has the advantage that it is relatively simple to define and, indeed, over the rational numbers it is equivalent to the more general theory. However, commutative DGAs only have good homotopy theory when we work over the rational numbers; over the integers or in characteristic $p$ a more rigid theory is needed. In his work on the cotangent complex, Illusie [**33**] worked with schemes with sheaves of simplicial commutative algebras. This was also the point of view of Lurie in his thesis [**45**]. However, there are basic examples arising from homotopy theory which cannot come from simplicial algebras – complex $K$-theory is an important example. Thus, a derived scheme (or stack) will be a scheme equipped with a sheaf of commutative ring spectra. I immediately remark that "commutative" is a difficult notion to define in ring spectra: here I mean "$E_\infty$-ring spectra". The foundations of commutative ring spectra are forbidding, but I'll make some attempt at an exposition below in Section 2.1.

A spectrum $X$ has homotopy groups $\pi_k X$, for $k \in \mathbb{Z}$. If $X$ is a commutative ring spectrum, $\pi_0 X$ is a commutative ring and the graded abelian group $\pi_* X$ is a graded skew-commutative $\pi_0 X$-algbera. In particular, $\pi_k X$ is a $\pi_0 X$-module.

Definition 1.1. — *A* derived scheme $(X, \mathcal{O})$ *is a pair with $X$ a topological space and $\mathcal{O}$ a sheaf of commutative ring spectra on $X$ so that*

(1) *the pair $(X, \pi_0 \mathcal{O})$ is a scheme; and*
(2) *the sheaf $\pi_k \mathcal{O}$ is a quasi-coherent sheaf of $\pi_0 \mathcal{O}$-modules.*

It is somewhat subtle to define the notion of a sheaf of spectra; I will come back to that point below in Remark 2.7. One definition of "quasi-coherent" is to require the sheaf to be locally the cokernel of a morphism between locally free sheaves. On the spectrum of a ring $R$, these are the module sheaves which arise from $R$-modules. There is a completely analogous definition of a derived Deligne-Mumford stack, except that now we must be careful about the topology we use to define sheaves: for these objects we must use the étale topology.

There is a rich structure inherent in Definition 1.1. The homotopy groups of a commutative ring spectrum support far more structure than simply that of a graded skew-commutative ring; in particular, it is a ring with "power operations". See Remark 2.6 below. Thus, if $(X, \mathcal{O})$ is a derived scheme, the graded sheaf $\pi_*\mathcal{O}$ is a sheaf of graded rings with all of this higher order structure.

### 1.3. Topological modular forms

On the compactified Deligne-Mumford stack $\bar{\mathcal{M}}_{ell}$, there is a canonical quasi-coherent sheaf $\omega$. If $C$ is an elliptic curve over $X$, then $C$ is an abelian variety over $X$ of relative dimension 1 and we can construct the sheaf of invariant 1-forms $\omega_C$ for $C$. This is a locally free sheaf of rank 1 on $X$ and the assignment

$$\omega(C : X \to \bar{\mathcal{M}}_{ell}) = \omega_C$$

defines a quasi-coherent sheaf on $\bar{\mathcal{M}}_{ell}$. The sheaf $\omega$ is locally free of rank 1, hence invertible, and the tensor powers $\omega^{\otimes k}$, $k \in \mathbb{Z}$, are all quasi-coherent. Here is the main result; see **[26]** and **[46]**.

THEOREM 1.2 (Hopkins-Miller-Lurie). — *There exists a derived Deligne-Mumford stack $(\bar{\mathcal{M}}_{ell}, \mathcal{O})$ so that*

(1) *the underlying algebraic stack $(\bar{\mathcal{M}}_{ell}, \pi_0\mathcal{O})$ is equivalent to the compactified Deligne-Mumford moduli stack $(\bar{\mathcal{M}}_{ell}, \mathcal{O}_{ell})$ of generalized elliptic curves; and*
(2) *there are isomorphisms of quasi-coherent sheaves $\pi_{2k}\mathcal{O} \cong \omega^{\otimes k}$ and $\pi_{2k+1}\mathcal{O} = 0$.*

*Furthemore, the derived stack $(\bar{\mathcal{M}}_{ell}, \mathcal{O})$ is determined up to equivalence by conditions* (1) *and* (2).

The doubling of degrees in (2) is quite typical: when the homotopy groups of a commutative ring spectrum $X$ are concentrated in even degrees, then $\pi_*X$ is a graded commutative ring, not just commutative up to sign.

We can rephrase the uniqueness statement of Theorem 1.2 as follows: there is a space of all derived stacks which satisfy points (1) and (2) and this space is path-connected. In fact, Lurie's construction gives a canonical base-point; that is, a canonical model for $(\bar{\mathcal{M}}_{ell}, \mathcal{O})$.

Here is a definition of the object in the title of this manuscript.