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GAUGE THEORY AND LANGLANDS DUALITY

by **Edward FRENKEL**

INTRODUCTION

In the late 1960s Robert Langlands launched what has become known as the Langlands Program with the ambitious goal of relating deep questions in Number Theory to Harmonic Analysis [**39**]. In particular, Langlands conjectured that Galois representations and motives can be described in terms of the more tangible data of automorphic representations. A striking application of this general principle is the celebrated Shimura–Taniyama–Weil conjecture (which implies Fermat's Last Theorem), proved by A. Wiles and others, which says that information about Galois representations associated to elliptic curves over $\mathbb{Q}$ is encoded in the Fourier expansion of certain modular forms on the upper-half plane.

One of the most fascinating and mysterious aspects of the Langlands Program is the appearance of the *Langlands dual group*. Given a reductive algebraic group G, one constructs its Langlands dual ${}^L G$ by applying an involution to its root data. Under the Langlands correspondence, automorphic representations of the group G correspond to Galois representations with values in ${}^L G$.

Surprisingly, the Langlands dual group also appears in Quantum Physics in what looks like an entirely different context; namely, the *electro-magnetic duality*. Looking at the Maxwell equations describing the classical electromagnetism, one quickly notices that they are invariant under the exchange of the electric and magnetic fields. It is natural to ask whether this duality exists at the quantum level. In quantum theory there is an important parameter, the electric charge e. Physicists have speculated that there is an electro-magnetic duality in the quantum theory under which $e \longleftrightarrow 1/e$.

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Under this duality the electrically charged particle should be exchanged with a magnetically charged particle, called magnetic monopole, first theorized by P. Dirac (so far, it has not been discovered experimentally).

In modern terms, Maxwell theory is an example of 4D *gauge theory* (or Yang–Mills theory) which is defined, classically, on the space of connections on various G_c-bundles on a four-manifold M, where G_c is a compact Lie group.(1) Electromagnetism corresponds to the simplest, abelian, compact Lie group $U(1)$. It is natural to ask whether there is a non-abelian analogue of the electro-magnetic duality for gauge theories with non-abelian gauge groups.

The answer was proposed in the late 1970s, by Montonen and Olive [**46**], following Goddard, Nuyts and Olive [**25**] (see also [**12, 50**]). A gauge theory has a coupling constant g, which plays the role of the electric charge e. The conjectural non-abelian electro-magnetic duality, which has later become known as S-*duality*, has the form

$$(G_c, g) \longleftrightarrow ({}^L G_c, 1/g). \tag{0.1}$$

In other words, the duality states that the gauge theory with gauge group G_c (more precisely, its "$N = 4$ supersymmetric" version) and coupling constant g should be equivalent to the gauge theory with the Langlands dual gauge group ${}^L G_c$ and coupling constant $1/g$ (note that if $G_c = U(1)$, then ${}^L G_c$ is also $U(1)$). If true, this duality would have tremendous consequences for quantum gauge theory, because it would relate a theory at small values of the coupling constant (weak coupling) to a theory with large values of the coupling constant (strong coupling). Quantum gauge theory is usually defined as a power series expansion in g, which can only converge for small values of g. It is a very hard problem to show that these series make sense beyond perturbation theory. S-duality indicates that the theory does exist non-perturbatively and gives us a tool for understanding it at strong coupling. That is why it has become a holy grail of modern Quantum Field Theory.

Looking at (0.1), we see that the Langlands dual group shows up again. Could it be that the Langlands duality in Mathematics is somehow related to S-duality in Physics?

This question has remained a mystery until about five years ago. In March of 2004, DARPA sponsored a meeting of a small group of physicists and mathematicians at the Institute for Advanced Study in Princeton (which I co-organized) to tackle this question. At the end of this meeting Edward Witten gave a broad outline of a relation between the two topics. This was explained in more detail in his subsequent joint work [**34**] with Anton Kapustin. This paper, and the work that followed it, opened new

(1) We will use the notation G for a complex Lie group and G_c for its compact form. Note that physicists usually denote by G a compact Lie group and by $G_{\mathbb{C}}$ its complexification.

bridges between areas of great interest for both physicists and mathematicians, leading to new ideas, insights and directions of research.

The goal of these notes is to describe briefly some elements of the emerging picture. In Sections 1 and 2, we will discuss the Langlands Program and its three flavors, putting it in the context of André Weil's "big picture". This will eventually lead us to a formulation of the geometric Langlands correspondence as an equivalence of certain categories of sheaves in Section 3. In Section 4 we will turn to the S-duality in topological twisted $N = 4$ super-Yang–Mills theory. Its dimensional reduction gives rise to the Mirror Symmetry of two-dimensional sigma models associated to the Hitchin moduli spaces of Higgs bundles. In Section 5 we will describe a connection between the geometric Langlands correspondence and this Mirror Symmetry, following [**34**], as well as its ramified analogue [**26**]. In Section 6 we will discuss subsequent work and open questions.

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1. LANGLANDS PROGRAM

In 1940 André Weil was put in jail for his refusal to serve in the army. There, he wrote a letter to his sister Simone Weil (a noted philosopher) in response to her question as to what really interested him in his work [**36**]. This is a remarkable document, in which Weil tries to explain, in fairly elementary terms (presumably, accessible even to a philosopher), the "big picture" of mathematics, the way he saw it. I think this sets a great example to follow for all of us.

Weil writes about the role of *analogy* in mathematics, and he illustrates it by the analogy that interested him the most: between Number Theory and Geometry.

On one side we look at the field $\mathbb{Q}$ of rational numbers and its algebraic closure $\overline{\mathbb{Q}}$, obtained by adjoining all roots of all polynomial equations in one variable with rational coefficients (like $x^2 + 1 = 0$). The group of field automorphisms of $\overline{\mathbb{Q}}$ is the *Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We are interested in the structure of this group and its finite-dimensional representations. We may also take a more general *number field*—that is, a finite extension F of $\mathbb{Q}$ (such as $\mathbb{Q}(i)$)—and study its Galois group and its representations.

On the other side we have Riemann surfaces: smooth compact orientable surfaces equipped with a complex structure, and various geometric objects associated to them: vector bundles, their endomorphisms, connections, etc.

At first glance, the two subjects are far apart. However, it turns out that there are many analogies between them. The key point is that there is another class of objects which are in-between the two. A Riemann surface may be viewed as the set of points of a projective algebraic curve over $\mathbb{C}$. In other words, Riemann surfaces may be described by algebraic equations, such as the equation

$$y^2 = x^3 + ax + b, \tag{1.1}$$

where $a, b \in \mathbb{C}$. The set of complex solutions of this equation (for generic a, b for which the polynomial on the right hand side has no multiple roots), compactified by a point at infinity, is a Riemann surface of genus 1. However, we may look at the equation (1.1) not only over $\mathbb{C}$, but also over other fields—for instance, over finite fields.

Recall that there is a unique, up to an isomorphism, finite field $\mathbb{F}_q$ of q elements for all q of the form p^n, where p is a prime. In particular, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \simeq \{0, 1, \ldots, p-1\}$, with the usual arithmetic modulo p. Let a, b be elements of $\mathbb{F}_q$. Then the equation (1.1) defines a curve over $\mathbb{F}_q$. These objects are clearly analogous to algebraic curves over $\mathbb{C}$ (that is, Riemann surfaces). But there is also a deep analogy with number fields!

Indeed, let X be a curve over $\mathbb{F}_q$ (such as an elliptic curve defined by (1.1)) and F the field of rational functions on X. This *function field* is very similar to a number field. For instance, if X is the projective line over $\mathbb{F}_q$, then F consists of all fractions $P(t)/Q(t)$, where P and Q are two relatively prime polynomials in one variable with coefficients in $\mathbb{F}_q$. The ring $\mathbb{F}_q[t]$ of polynomials in one variable over $\mathbb{F}_q$ is similar to the ring of integers and so the fractions $P(t)/Q(t)$ are similar to the fractions p/q, where $p, q \in \mathbb{Z}$.

Thus, we find a *bridge*, or a "turntable"—as Weil calls it—between Number Theory and Geometry, and that is the theory of algebraic curves over finite fields.

In other words, we can talk about three parallel tracks

Number Theory *Curves over* $\mathbb{F}_q$ *Riemann Surfaces*

Weil's idea is to exploit it in the following way: take a statement in one of the three columns and translate it into statements in the other columns [**36**]: "my work consists in deciphering a trilingual text; of each of the three columns I have only disparate fragments; I have some ideas about each of the three languages: but I know as well there are great differences in meaning from one column to another, for which nothing has prepared me in advance. In the several years I have worked at it, I have found little pieces of the dictionary." Weil went on to find one of the most spectacular