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(1013) *p-adic families of modular forms*

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p-ADIC FAMILIES OF MODULAR FORMS
[after Hida, Coleman, and Mazur]

by **Matthew EMERTON**

INTRODUCTION

The theory of p-adic families of modular forms grew out of two highly related traditions in the arithmetic theory of modular forms: the theory of congruences of modular forms (which dates back to work of Ramanujan) and the (more recent) theory of Galois representations attached to modular forms. The first example of a p-adic family of modular forms was the Eisenstein family, considered by Serre in [**37**]. This is a family of q-expansions, parametrized by the weight k, whose coefficients are p-adically continuous functions of k. Serre's immediate goal in studying this family was to obtain an understanding of the possible congruences between the q-expansion coefficients of modular forms in different weights, especially of the constant terms, since such congruences lead to congruences between special values of ζ-functions.

The papers [**23, 22**] led to a decisive shift in the theory, placing it at the centre of the arithmetic theory of modular forms. In these papers, Hida constructed p-adic families of cuspforms, varying continuously with the weight k, which were also simultaneous eigenforms for the Hecke operators. Thus, in light of the known construction of Galois representations attached to Hecke eigenforms, one found that associated to these p-adic families of cuspidal eigenforms there were corresponding p-adic families of p-adic Galois representations. The existence of such families led Mazur to develop his general theory of deformations of Galois representations [**31**], which in turn inspired further developments [**45, 43**].

Hida's constructions had a certain limitation: if f is a Hecke eigenform of weight $k \geq 1$ and level N prime to p, then f appears in a Hida family if and only if (at least) one of the roots of the pth Hecke polynomial of f is of slope zero (i.e. a p-adic unit). This restriction was removed by the work of Coleman and Mazur [**10**], who constructed p-adic analytic (more precisely, rigid analytic) curves of eigenforms containing any such form f, whether or not its pth Hecke polynomial admits a unit root; these are the so-called *eigencurves*.

The eigencurves are fundamentally analytic objects. One can also ask whether there is an algebraic family (or more precisely, a scheme) that parametrizes all the f as above, regardless of the slopes of the roots of the pth Hecke polynomial. Indeed, there is such an object; all the eigenforms f (of arbitrary weight but some fixed level N) are parametrized by the $\overline{\mathbb{Z}}_p$ points of $\operatorname{Spec} \mathbb{T}(N)$, where $\mathbb{T}(N)$ is the p-adic Hecke algebra of level N. These points are no longer parametrized by weight; indeed, $\operatorname{Spec} \mathbb{T}(N)$ is (at least conjecturally) of relative dimension three over $\operatorname{Spec} \mathbb{Z}_p$. It is conjectured that every continuous, two-dimensional, semi-simple odd p-adic Galois representation of $G_{\mathbb{Q}}$ that is unramified outside finitely many primes corresponds to a point of $\operatorname{Spec} \mathbb{T}(N)$ for some appropriate value of N. This is one of the main motivations for the study of the families $\operatorname{Spec} \mathbb{T}(N)$, and the related p-adic families of eigenforms constructed by Hida and Coleman–Mazur.

In Section 1 of this exposé we recall the basic theory of modular forms, Hecke operators, and the Galois representations associated to Hecke eigenforms. In Section 2, we outline the definitions and basic results and conjectures regarding the p-adic Hecke algebras $\mathbb{T}(N)$, and the families of Hida and Coleman–Mazur. We focus more on systems of Hecke eigenvalues attached to eigenforms, rather than on the eigenforms themselves. This is in keeping with our focus on the relationship with Galois representations (although it takes us somewhat far in spirit from the concrete viewpoint of [**37**]).

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0.1. Notation

As usual $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational, real, and complex numbers, and $\mathbb{Z}$ denotes the ring of integers. For any prime p, we let $\mathbb{Z}_p$ denote the ring of p-adic integers, and $\mathbb{Q}_p$ denote the field of p-adic numbers.

We let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let $\overline{\mathbb{Z}}$ denote the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$. For each prime p, we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and let $\overline{\mathbb{Z}}_p$ denote the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$. We also fix an embedding $\imath_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. This restricts to an embedding $\overline{\mathbb{Z}} \hookrightarrow \overline{\mathbb{Z}}_p$. We write $\overline{\mathbb{F}}_p$ to denote the residue field of $\overline{\mathbb{Z}}_p$. It is an algebraic closure of the field $\mathbb{F}_p$ of p elements. We let $\operatorname{ord}_p : \overline{\mathbb{Q}}_p \to \mathbb{Z} \cup \{\infty\}$ denote the p-adic valuation, normalized so that $\operatorname{ord}_p(p) = 1$. If $x \in \overline{\mathbb{Q}}_p$, then $\operatorname{ord}_p(x)$ is also called the *slope* of x. (Thus x has finite slope if and only if $x \neq 0$, while x has slope zero if and only if $x \in \overline{\mathbb{Z}}_p^{\times}$.)

1. MODULAR FORMS, HECKE ALGEBRAS, AND GALOIS REPRESENTATIONS

1.1. Modular forms

Let

$$\mathscr{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$$

denote the complex upper half-plane. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathscr{H}$ in the usual way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

Let $\mathscr{O}(\mathscr{H})$ denote the space of holomorphic functions on $\mathscr{H}$. If k is an integer, then we define the weight k-action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathscr{O}(\mathscr{H})$ as follows:

$$(f \mid_k \gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau),$$

for $f \in \mathscr{O}(\mathscr{H})$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$; as the notation indicates, this is a right action.

If $N \geq 1$, define

$$\Gamma_1(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \bmod N\right\}.$$

Definition 1.1. — *A modular form (resp. cuspform) of weight k and level N is a holomorphic function $f \in \mathscr{O}(\mathscr{H})$ that is invariant under the weight k-action of $\Gamma_1(N)$, and for which*

$$\lim_{y \to \infty} (f \mid_k \gamma)(iy) \tag{1}$$

exists and is finite (resp. vanishes) for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. We let $\mathscr{M}_k(N)$ (resp. $\mathscr{S}_k(N)$) denote the space of modular forms (resp. cuspforms) of weight k and level N.

Remark 1.2. — If $f \in \mathscr{O}(H)$ is invariant under the weight k-action of $\Gamma_1(N)$, then, in order to check if f is a modular form or a cuspform, it suffices to study the limit (1) for finitely many $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ (namely, for a set of coset representatives for $\Gamma_1(N)\backslash\mathrm{SL}_2(\mathbb{Z})$).

Remark 1.3. — If f is a modular form of weight k and level N, then, applying the invariance property of f to the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_1(N)$, one finds that $f(\tau+1) = f(\tau)$. We may thus expand the function $f(\tau)$ as a Fourier series

$$f(\tau) := \sum_{n=-\infty}^{\infty} c_n(f) q^n,$$

where $q := \exp(2\pi i\tau)$. Condition (1), with $\gamma = 1$, then shows that $c_n(f) = 0$ for $n < 0$ (resp. for $n \leq 0$ if f is a cuspform). We refer to this Fourier series as the q-expansion of f.

Clearly $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$ are vector subspaces of $\mathcal{O}(\mathcal{H})$. In fact they are also finite dimensional. (See **[39]** for a discussion of this and other basic facts concerning modular forms.)

Example 1.4. — If $k < 0$, then $\mathcal{M}_k(N) = 0$. When $k = 0$, the space $\mathcal{M}_0(N)$ consists simply of the constant functions on $\mathcal{H}$ (and so $\mathcal{S}_0(N) = 0$). To avoid these trivial cases, we will typically assume that $k \geq 1$ in all that follows. As k increases, the dimensions of both $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$ grow essentially linearly in k (with the exception that $\mathcal{M}_k(N) = 0$ if $N = 1$ or 2 and k is odd).

Example 1.5. — The simplest examples of modular forms of positive weight are the Eisenstein series $E_k \in \mathcal{M}_k(1)$. These are defined for even $k \geq 4$. (It is easily shown that $\mathcal{M}_k(1)$ vanishes if k is odd or $0 < k < 4$.) The q-expansion of E_k is given by the following formula:

$$E_k(\tau) = \frac{-B_k}{2k} + \sum_{n=1}^{\infty}\Big(\sum_{d|n} d^{k-1}\Big)q^n,$$

where B_k is the kth Bernoulli number.

There is a direct sum decomposition

$$\mathcal{M}_k(1) = \mathbb{C}E_k \oplus \mathcal{S}_k(1).$$

More generally, for any N, we may decompose $\mathcal{M}_k(N)$ into the direct sum of a space of Eisenstein series (typically of dimension greater than one when $N > 1$) and the space of cuspforms. (See Example 1.18 below.)

1.2. Hecke operators

Fix integers $k \geq 1$ and $N \geq 1$. Write

$$\Gamma_0(N) := \big\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \,|\, \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod N\big\}.$$

Note that $\Gamma_0(N)$ contains $\Gamma_1(N)$ as a normal subgroup, and that the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \bmod N$$

induces an isomorphism

$$\Gamma_0(N)/\Gamma_1(N) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{\times}. \tag{2}$$

A simple computation, using the normality of $\Gamma_1(N)$ in $\Gamma_0(N)$, shows that the weight k-action of $\Gamma_0(N)$ preserves $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$. When restricted to these spaces, this action obviously factors through the quotient $\Gamma_0(N)/\Gamma_1(N)$, and hence, via the isomorphism (2), we obtain an action of the group $(\mathbb{Z}/N\mathbb{Z})^{\times}$ on $\mathcal{M}_k(N)$ and $\mathcal{S}_k(N)$. If $d \in (\mathbb{Z}/N\mathbb{Z})^{\times}$, then we denote the corresponding automorphism of $\mathcal{M}_k(N)$ by $\langle d\rangle$. (These operators are sometimes referred to as the diamond operators.)