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LINEAR STABILITY OF BLACK HOLES [d'après M. Dafermos et I. Rodnianski]

by **Sergiu KLAINERMAN**

The treatment of perturbations of Kerr spacetime has been prolixious in its complexity. Perhaps at a later time the complexity will be unravelled by deeper insights. But meantime the analysis has led into a realm of the rococo, splendorous, joyful and immensely ornate.

S. Chandrasekhar, *The mathematical theory of black holes.*

1. INTRODUCTION

While the splendorous remains, a layer of complexity has now been unravelled. I report on the recent, remarkable, ongoing progress made on the linear stability of black holes, more precisely on the boundedness and decay properties of solutions to linear equations in a Kerr spacetime. The Kerr spacetimes $\mathcal{K}(m,a)$ are explicit solutions of the Einstein vacuum equations (discovered by R. Kerr in 1963) depending on two parameters $0 \leq a \leq m$, corresponding physically to black holes of mass m and angular momentum am. The case $a = m = 0$ corresponds to the Minkowski space while $a = 0, m > 0$, corresponds to the much older Schwarzschild solution (K. Schwarzschild 1915).

The problem of linear stability of the Kerr family is an old problem which has received a lot of attention in the Physics literature immediately after the discovery of these fascinating solutions of the Einstein equations in vacuum, which, embedded in the larger 3-parameter family of the so-called Kerr-Newman spacetimes, form the basis of our understanding of black holes. The obvious question raised by the discovery of any interesting, explicit solution of a complex, non-linear system, such as the Einstein equations, is that of their stability under small perturbations. Roughly the problem here is to show that all spacetime developments of initial data sets, sufficiently close to the initial data set of a Kerr spacetime, behaves in the large like (possible another) Kerr solution. This is not only a deep mathematical question but one with serious astrophysical implications. Indeed if the Kerr family would be unstable under perturbations, black holes would be nothing more than mathematical artifacts. The Einstein equations are, of course, nonlinear and hyperbolic, thus the issue of stability is an extremely difficult and a dicey one. Given the geometric, covariant structure of the equations, with no universal notions of space and time variables, it is not even a-priori clear what that means. Linear stability, though still tricky, is somewhat easier to define. It is clear, for example, that any first order approximation of the equations, at the level of the space-time metric, in any reasonable coordinate system, will generate some system of wave equations in the Kerr background we want to perturb. Thus it is natural to ask, and this must certainly be relevant to the full nonlinear problem, whether solutions to linear wave equations in a fixed Kerr background are well behaved. If it turns out that solutions of these linear equations are amplified, due to the non-trivial features of the background geometry, then there is a reasonable chance that the background itself might be unstable.

It is not enough, however, to establish that solutions are not amplified; to have a chance to prove non-linear stability we also need to show that solutions decay at a sufficient rate. There is a lot of confusion in this regard among some physicists who seem to believe that somehow the lack of linear instability is a strong indication of nonlinear stability. This, of course, is not true even near solutions of minimal energy of simple nonlinear PDE's, as the case of the Burger equation $\partial_t u + u\partial_x u = 0$ easily demonstrates. The solution $u = 0$ is a global minimum for the energy integral $E(t) = \int |u(t,x)|^2 dx$, yet any compactly supported, smooth, small perturbation of the zero initial leads to blow up in finite time.

To be useful, a result on linear stability has to establish, *quantitatively*, not just a lack of amplification but also a realistic decay. In fact all known stability results, for strongly nonlinear wave equations (Einstein equations are quasilinear), depend on precise decay information for the linearized solutions.

The methods by which one establishes these decay estimates are also a very important issue. Thus, in the Minkowski space $\mathbb{R}^{1+3}$, it is easy to derive decay estimates for solution to the standard wave equation $\square\phi = 0$ using explicit representation formulas in the physical or Fourier variables. These formulas, however, depend heavily on the specific features of the Minkowski space and do not survive under small perturbations of the Minkowski metric. In other words, such methods are intrinsically not *robust*. A far more useful method for deriving decay estimates for the wave equation, and more generally for linear field equations, is that of *invariant vector fields*, see [**28**], [**29**]. That method, first introduced to prove stability results for quasilinear wave equations, plays a fundamental role in all known proofs of the stability of the Minkowski space, see [**12**], [**30**], [**32**], [**5**].

In the case of the Kerr metric, or rather the more accessible case of the Schwarzschild metric, one can use the specific symmetries of the space to separate variables and then concentrate on the pointwise properties of the corresponding eigenvalue problem. This method is not only not robust but, to our knowledge, was not even satisfactory to derive unconditional decay results for general solutions of the wave equation. In the physics literature, where the problem of linear stability of Schwarzschild and Kerr spacetimes has received a tremendous amount of attention (see e.g. [**40**], [**45**], [**39**], [**38**], [**41**], the monograph [**9**] and the references therein), this method of mode decomposition led to nothing more, in the words of Press and Teukolsky (see [**38**]), than *"an unsuccessful search for instabilities"*. On the other hand mathematical rigorous efforts based on this approach can only lead to statements of decay without a rate or precise rates of decay of specific modes, both of which, in principle, compatible with the scenario in which a general solution of the corresponding linear problem is not even uniformly bounded. For the results in this direction, see [**34**], [**31**], [**21**] in Schwarzschild and an attempt [**22**] in Kerr. Moreover, even if ultimately successful, such methods would leave us with a heavy machinery to prove some form of linear stability without any clue on how to approach to the non-linear problem.

A simple version of the vector field method was first used by Kay and Wald, see [**27**], to prove the boundedness of solutions of the wave equation in a Schwarzschild spacetime. The first attempt to use the vector field method, to prove integrated local energy decay in Schwarzschild is due to Blue and Soffer [**6**]. Their work however had serious flaws. The first complete results on pointwise decay for solutions of the wave equation on the Schwarzschild background have been obtained, independently, by Blue-Sterbenz [**7**], and Dafermos-Rodnianski [**16**]. In [**16**] Dafermos and Rodnianski also introduced the crucial red shift vector field, which led to stronger decay rates along the event horizon in Schwarzschild and, more importantly, played a central role in extending the boundedness and decay results to Kerr space-times, see [**20**], [**17**]. Other important contributions were made by S. Alinhac in [**3**], Dafermos-Rodnianski in [**19**], Marzuola-Metcalfe-Tataru-Tohaneanu in [**35**] and Luk in [**33**] for the problem in Schwarzschild, and by Tataru-Tohaneanu in [**43**] and Andersson-Blue in [**4**], for Kerr spacetimes.

I will review these results following, mainly the works of Dafermos-Rodnianski, in particular their general exposition in [**17**] and the recent paper [**18**].

2. INITIAL VALUE PROBLEM

We recall that an initial data set consists of a 3-dimensional manifold Σ, a complete Riemannian metric $g_{(0)}$, a symmetric 2-tensor $k_{(0)}$, and a well specified set of initial conditions corresponding to the matterfields under consideration. These have to be restricted to a well known set of constraint equations. We restrict the discussion to *asymptotically flat* initial data sets, i.e. outside a sufficiently large compact set K, $\Sigma_{(0)} \setminus K$ is diffeomorphic to the complement of the unit ball in $\mathbb{R}^3$ and admits a system of coordinates in which $g_{(0)}$ is asymptotically euclidean and $k_{(0)}$ vanishes, at appropriate order. A *Cauchy development* of an initial data set is a globally hyperbolic spacetime $(\mathcal{M}, \mathbf{g})$, verifying the Einstein field equations, in the presence of a matterfield with energy momentum $\mathbf{Q}$,

$$\mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\alpha\beta} = \mathbf{Q}_{\alpha\beta}, \tag{2.1}$$

and an embedding $i : \Sigma \longrightarrow \mathcal{M}$ such that $i_*(g_{(0)}), i_*(k_{(0)})$ are the first and second fundamental forms of $i(\Sigma_{(0)})$ in $\mathcal{M}$.

In what follows I will restrict the discussion to the Einstein vacuum equations, i.e. the case when the energy momentum tensor vanishes identically and the equations take the purely geometric form

$$\mathbf{R}_{\alpha\beta} = 0. \tag{2.2}$$

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FIGURE 1.

The most primitive question asked about the initial value problem, solved in a satisfactory way, for very large classes of evolution equations, is that of local existence and uniqueness of solutions. For the Einstein equations this type of result was first