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ERGODICITY OF TWO DIMENSIONAL TURBULENCE [after Hairer and Mattingly]

by **Antti KUPIAINEN**

INTRODUCTION

The problem of turbulence has been described as the last great unsolved problem of classical physics. Understanding of the complicated motion of fluids in the presence of obstacles or stirring has been a challenge to mathematicians, physicists and engineers for quite a time now. The equations governing macroscopic fluid motion, the Navier Stokes equations, have been known for close to two centuries. For an incompressible fluid in units where the density equals one they read

$$\partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p + f. \tag{1}$$

$u(t,x) \in \mathbb{R}^d$ is the velocity field at time t at $x \in \Lambda$, a domain in $\mathbb{R}^d$ subject to the incompressibility condition

$$\nabla \cdot u = 0 \tag{2}$$

and suitable boundary conditions on $\partial\Lambda$. ν is the viscosity coefficient of the fluid, $p(t,x)$ the pressure and $f(t,x)$ the external force that sustains the flow. Given f and $u(0,\cdot)$ the task is to find u and p. It is fair to say that theoretical understanding of the consequences of these equations is still in its infancy. On the mathematical side, existence of smooth solutions for the three dimensional NS equations is wide open and has been chosen by some as one of the major problems of mathematics (`http://www.claymath.org/millennium/`). On the physical side, experimental violations of the Kolmogorov scaling theory of turbulence [**12**] are still waiting for theoretical understanding.

In two dimensions, i.e., for flows on the plane, there has been some progress during the last ten years. On the physical side, 2d turbulence has been the subject of accurate numerical and experimental studies [**5**], [**25**] and mathematically the ergodic theory of the NS flow has been under intensive study.

It is important to realize that for the problem of turbulence one is interested in a very particular kind of force in (1), namely one that has a fixed length scale L built into it. Examples of this are flows past obstacles, with L the characteristic size of the obstacle. In such a setup the flow exhibits universal statistical properties as the viscosity parameter tends to zero (actually the control parameter is a dimensionless quantity, the Reynolds number given by $\frac{Lv}{\nu}$ where v is a velocity scale related to the forcing). E.g. time averages of measurements of suitable functions of u seem to show statistical properties only depending on the Reynolds number. It is therefore of some interest to inquire about the foundations for such statistical studies, i.e., about the ergodic properties of the NS flow in the turbulent setup of a fixed scale high Reynolds number forcing.

A convenient model for isotropic and homogeneous turbulence (i.e., in the limit of large Reynolds number and away from the boundary $\partial\Lambda$) is to consider [(1)] Equation (1) on the torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ and take f random, a Fourier series with a finite number of terms and coefficients independent white noises (see below). Then the deterministic dynamics of (1) is replaced by a Markov process and one may pose questions on its ergodic properties: whether the process has a unique stationary state and whether this is reached and with what rate from arbitrary initial conditions.

This Markov process is a diffusion process of a very degenerate type. While the phase space is infinite dimensional the noise is finite dimensional. There are two general mechanisms that can contribute to the ergodic and mixing properties of stochastic flows. One is dissipation, coming in our case from the Laplacian in (1). Dissipation contributes to ergodicity by exponential contraction of phase space under the flow. A second mechanism comes from the spreading of the noise from its finite dimensional subspace due to the nonlinear term in (1). In finite dimensional diffusion processes this leads to hypoellipticity if the noise spreads to the full phase space: the transition kernels are smooth (for equations with smooth coefficients). Combined with some irreducibility of the process ergodicity follows.

In our infinite dimensional setup the dissipation due to the Laplacian leads to strong damping of large enough (depending on the Reynolds number) Fourier modes. If we keep noise on all the other, low, modes then one can reduce the problem to a low mode dynamics, albeit with some (exponentially decaying) memory due to the large modes. Proofs of ergodicity and mixing of the dynamics were given in this case in the works [**6**], [**10**] and [**18**]. However, it seemed far from trivial to extend the hypoellipticity ideas to the infinite dimensional setup to control also the case of very

(1) To get to the turbulent state one actually has to modify (1) a bit, see Section 8.

degenerate forcing where the number of forced modes does not depend on the Reynolds number. This was accomplished by Hairer and Mattingly [**13**], [**15**] who gave sharp sufficient conditions for the noise to produce ergodic and mixing dynamics. In what follows I will present the main points of their approach focusing on the difference to finite dimensional hypoelliptic diffusions. The papers [**13**], [**15**] are very clearly written and they contain plenty of background material, especially [**15**] which builds a more general formalism applicable also to some reaction-diffusion equations. [**15**] also corrects a mistake in [**13**] so it should be consulted for a thorough study. In the final section I discuss more informally what we have learned about 2d turbulence and what issues might be accessible to a rigorous mathematical analysis.

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1. 2D NS EQUATIONS

The fundamental fact that is behind both the mathematical and physical understanding of 3d NS equations is energy conservation: in the absence of forces smooth inviscid flow preserves the L^2 norm of $u(t,\cdot)$. In two dimensions there is a second conserved quantity, the *enstrophy*, which is related to the H^1 norm and which leads to quite different physics and to much better regularity.

Let us first define the *vorticity*

$$\omega = \nabla \times u,$$

which in $d = 2$ is a (pseudo)scalar: $\omega = \partial_1 u_2 - \partial_2 u_1$. The NS equation becomes in terms of ω a transport equation:

$$\dot{\omega} = \nu \Delta \omega - u \cdot \nabla \omega + g, \tag{3}$$

where $g = \partial_1 f_2 - \partial_2 f_1$. We will assume the average force vanishes, i.e., $\int f(t,x)dx = 0$. Then (1) preserves the condition $\int u(t,x)dx = 0$ which we will assume. The incompressibility condition (2) allows to write $u = \mathcal{C}\omega$ where the linear operator $\mathcal{C}$ is given in terms of the Fourier transform by

$$\widehat{\mathcal{C}\omega}(k) = i(k_2, -k_1)k^{-2}\hat{\omega}(k) \tag{4}$$

for $k \in \mathbb{Z}^2 \setminus 0$.

The enstrophy $\mathcal{E}$ is defined to be (half of) the L^2-norm of ω:

$$\mathcal{E} = \tfrac{1}{2}\int \omega(t,x)^2 dx := \tfrac{1}{2}\|\omega(t)\|^2.$$

For a smooth u the condition $\nabla \cdot u = 0$ leads to the absence of contribution from the nonlinear term to the evolution of the enstrophy:

$$\frac{d\mathcal{E}}{dt} = -\nu \int (\nabla \omega)^2 dx + \int \omega g dx, \tag{5}$$

where the first term on the RHS can be interpreted as an enstrophy dissipation rate and the second one as an enstrophy injection rate. Using Poincaré inequality $\|\nabla \omega\| \geq \|\omega\|$ and simple estimates one deduces

$$\|\omega(t)\|^2 \leq e^{-\nu t}\|\omega(0)\|^2 + \nu^{-2} \sup_t \|g(t)\|^2. \tag{6}$$

This a priori estimate for the H^1 norm of u is the main ingredient in the proof of global regularity of the 2d NS flow.

We wish now to discuss a version of (3) where the force g is random. We work in the subspace of real valued $L^2(\mathbb{T}^2)$ functions with $\hat{\omega}(0) = 0$. It will be convenient to use the following basis for this space. Let Z^+ be the "upper half plane" in $\mathbb{Z}^2$ consisting of $k = (k_1, k_2)$ with $k_2 > 0$ or $k_2 = 0$ and $k_1 > 0$. Hence $\mathbb{Z}^2 \setminus 0 = Z^+ \cup (-Z^+)$. Let $e_k = \sin kx$ for $k \in Z^+$ and $e_k = \cos kx$ for $-k \in Z^+$. For each $k \in \mathbb{Z}^2$ pick independent Brownian motions $\beta_k(t)$ with unit speed, denoted collectively by $\beta(t)$ and numbers $\gamma_k \in \mathbb{R}$. Let

$$Q\beta(t) = \sum_{k \in \mathbb{Z}^2} \gamma_k \beta_k(t) e_k. \tag{7}$$

The stochastic version of Equation (3) reads

$$d\omega = (\nu \Delta \omega - u \cdot \nabla \omega)dt + Qd\beta. \tag{8}$$

Regularity of the stochastic flow proceeds in parallel with the deterministic case as long as γ_k have enough decay at infinity. The analog of the enstrophy conservation Equation (5) is obtained by an application of the Ito formula

$$d\mathcal{E} = \tfrac{1}{2} d\|\omega\|^2 = -\nu \|\nabla \omega\|^2 dt + (\omega, Qd\beta) + \epsilon dt \tag{9}$$

where $\epsilon = 2\pi^2 \sum_k \gamma_k^2$ can be interpreted as the enstrophy injection rate. Taking averages we get a probabilistic analog of (5) and (6):

$$\frac{d}{dt}\mathbb{E}\mathcal{E} = -\nu \mathbb{E}\|\nabla \omega\|^2 + \epsilon \tag{10}$$

and

$$\mathbb{E}\|\omega(t)\|^2 \leq e^{-2\nu t}\|\omega(0)\|^2 + \nu^{-1}\epsilon. \tag{11}$$

Actually (9) can be used to control exponential moments of the enstrophy **[6]**, **[13]** Lemma A.1:

$$\mathbb{E} \exp(\eta \|\omega(t)\|^2) \leq 2 \exp(\eta e^{-\nu t} \|\omega(0)\|^2) \tag{12}$$