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(1022) *Fundamental groups of Kähler manifolds and geometric group theory*

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FUNDAMENTAL GROUPS OF KÄHLER MANIFOLDS AND GEOMETRIC GROUP THEORY

by **Marc BURGER**

INTRODUCTION

The aim of this note is to report on some recent progress in the problem of characterizing fundamental groups of compact Kähler manifolds, henceforth called Kähler groups. More precisely we will illustrate, by means of a specific result, the program outlined by Delzant and Gromov in [**16**]: "Identify the constraints imposed by the Kähler nature of the space on the asymptotic invariants of its fundamental group and then express these invariants in terms of algebraic properties".

The result we have in mind is the theorem of T. Delzant [**15**] which says that a solvable Kähler group contains a nilpotent subgroup of finite index. This is based on the explicit description of the Bieri–Neumann–Strebel invariant of a Kähler group $\pi_1(M)$ in terms of factorizations of M over hyperbolic Riemann surfaces.

Before we come to this main topic we will recall what a Kähler manifold is, then list in telegraphic style results giving restrictions on Kähler groups and give a series of examples. For a more complete account of the theory of Kähler groups up to 1995, see [**1**].

Let M be a complex manifold with a Hermitian metric h, that is a collection of Hermitian metrics h_x on each tangent space T_xM, varying smoothly with x. Then the real part $g := \Re h$ gives a Riemannian metric on the underlying real manifold and the imaginary part $\omega := \Im h$ gives a real two-form. Together with the complex structure J we have

$$\omega(X,Y) = g(X,JY)\,. \tag{1}$$

The Hermitian manifold (M,h) is Kähler if $d\omega = 0$. An elementary consequence of this relation is that at each point of M there exist holomorphic coordinates such that the Hermitian metric equals the flat metric on $\mathbf{C}^n$ up to and including terms of first order, [**43**, 3.14]. This readily implies the Kähler identities ([**43**, 6.1]) which are at the basis of the Hodge decomposition of the cohomology of compact Kähler manifolds. The compatibility condition (1) implies that $\omega^n = n!d\,\mathrm{vol}_g$, in particular ω is non-degenerate at each point, i. e. it is a symplectic form and, when M is compact, defines therefore a non-zero class in $\mathrm{H}^2(M, \mathbf{R})$.

The following two observations lead to an important class of Kähler manifolds:

- the induced Hermitian structure on a complex submanifold $N \subset M$ of a Kähler manifold is Kähler;
- up to a positive multiple, there is a unique $\mathrm{SU}(n+1)$-invariant Hermitian metric on $\mathbf{CP}^n$; since its imaginary part ω is an invariant two-form, it is closed. Normalizing the metric so that $\int_{\mathbf{CP}^1} \omega = 1$, one obtains the Fubini–Study metric.

Thus every smooth projective manifold is a Kähler manifold. In our context this leads to the question whether every Kähler group is also the fundamental group of a smooth projective variety, to which we do not know the answer. Remarkably, concerning homotopy type, we have, thanks to Voisin [**44**], examples of compact Kähler manifolds which do not have the homotopy type of a smooth projective variety. Finally, it is a natural question whether the existence of a complex structure and/or a symplectic structure on a compact manifold imposes additional restrictions on its fundamental group, beyond being finitely presentable. In fact, every finitely presentable group is the fundamental group of a complex threefold which is also symplectic ([**18**], see also [**30**, 7.2]); it is thus the compatibility between these two structures, that is the defining property of a Kähler structure, which will give restrictions on its fundamental group.

1. RESTRICTIONS

In this section $\Gamma = \pi_1(M)$ is the fundamental group of a compact Kähler manifold M with Kähler form ω.

1.1. The first Betti number $b_1(\Gamma)$ is even

The vector space $\mathrm{Hom}(\Gamma, \mathbf{R}) = \mathrm{H}^1(\Gamma, \mathbf{R})$ is isomorphic to the space $\mathscr{H}^1(M)$ of real harmonic 1-forms on M; precomposition of 1-forms with J gives a complex structure on $\mathscr{H}^1(M)$ and hence its dimension $b_1(\Gamma)$ is even.

1.2. There is a non-degenerate skew-structure on $\mathrm{H}^1(\Gamma, \mathbf{R})$

On $\mathrm{H}^1(\Gamma, \mathbf{R})$ the form $(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta \wedge \omega^{n-1}$ is skew-symmetric and non-degenerate (Hard Lefschetz Theorem). Noting that the classifying map $M \to B\Gamma$ induces in cohomology an isomorphism in degree 1 and an injection in degree 2 shows that this skew-symmetric form factors through the cup product

$\Lambda^2 \mathrm{H}^1(\Gamma, \mathbf{R}) \to \mathrm{H}^2(\Gamma, \mathbf{R})$, which is therefore not zero if $b_1(\Gamma) > 0$, [**24**]. In this context there is a conjecture of Carlson and Toledo, namely that if Γ is infinite, there is $\Gamma' < \Gamma$ of finite index with $b_2(\Gamma') > 0$. For more on this, see [**29**, 18.16], [**26**], [**27**], [**28**].

1.3. The Malcev Lie algebra $\mathscr{L}\Gamma$ of Γ is quadratically presented

Associated to Γ there is a tower of nilpotent Lie algebras

$$\cdots \longrightarrow \mathscr{L}_n\Gamma \longrightarrow \mathscr{L}_{n-1}\Gamma \longrightarrow \cdots,$$

say over $\mathbf{R}$, where $\mathscr{L}_n\Gamma$ is the Lie algebra of the $\mathbf{R}$-unipotent algebraic group determined by the quotient $\Gamma/\mathscr{C}^n\Gamma$, where $\mathscr{C}^n\Gamma$ is the n-term of the descending central series. "Quadratic presentation" then means loosely that this tower of Lie algebras is determined by the map $\mathrm{H}_2(\Gamma) \to \Lambda^2 \mathrm{H}_1(\Gamma)$ (see [**1**, Chap. 3] and references therein).

1.4. A Kähler group has zero or one end(1)

The ideas and methods introduced by Gromov [**21**] leading to this result have been very influential in this field in the last twenty years. Here are some highlights. Recall that for the number $e(\Gamma)$ of ends of a finitely generated group we have $e(\Gamma) \in \{0, 1, 2, \infty\}$, with $e(\Gamma) = 0$ precisely when Γ is finite and $e(\Gamma) = 2$ precisely when Γ is virtually $\mathbf{Z}$; then Stallings' theorem says that $e(\Gamma) = \infty$ precisely when Γ is a nontrivial amalgam or an HNN-extension, both over a finite group. This theorem will however not be used in the proofs. The first step, which has nothing to do with Kähler manifolds, is the following

Proposition 1.1. — *If $\Gamma = \pi_1(M)$, where M is a compact Riemannian manifold and $e(\Gamma) = +\infty$, then the space $\mathscr{H}^1_{(2)}(\widetilde{M})$ of square integrable harmonic 1-forms on $\widetilde{M}$ is non-trivial, and in fact infinite dimensional. In particular, the reduced L^2-cohomology group $\overline{\mathrm{H}}^1(\Gamma, \ell^2(\Gamma))$ does not vanish, as it is isomorphic to $\mathscr{H}^1_{(2)}(\widetilde{M})$ by a variant of Dodziuk's de Rham theorem.*

The central result is then the following factorization theorem:

Theorem 1.2 ([**3**]). — *Let X be a complete Kähler manifold with bounded geometry and $\mathrm{H}^1(X, \mathbf{R}) = 0$. Assume that $\mathscr{H}^1_{(2)}(X) \neq 0$. Then there exists a proper holomorphic map with connected fibers $h : X \to \mathbb{D}$ to the Poincaré disk; moreover the fibers of h are permuted by* $\mathrm{Aut}(X)$.

We obtain then the following purely group theoretical consequence:

(1) A general reference for this section is [**1**, Ch. 4].

Corollary 1.3 ([**3**], [**21**]). — *Let* Γ *be a Kähler group with* $\overline{\mathrm{H}}^1(\Gamma, \ell^2(\Gamma)) \neq 0$. *Then* Γ *is commensurable to the fundamental group* Γ_g *of a compact orientable surface of genus* $g \geq 2$.

More precisely there are a subgroup $\Gamma' < \Gamma$ of finite index and an exact sequence

$$1 \longrightarrow F \longrightarrow \Gamma' \longrightarrow \Gamma_g \longrightarrow 1$$

with F finite. In particular $e(\Gamma) = e(\Gamma') = e(\Gamma_g) = 1$ and thus a Kähler group has zero or one end.

The factorization Theorem 1.2 follows from a general stability theorem for compact leaves in singular holomorphic foliations, which also plays a central role in the work of Delzant and Gromov on "Cuts in Kähler groups", [**16**] (see also § 1.5). Recall that the singular holomorphic foliation $\mathcal{F}_\eta$ associated to a closed holomorphic 1-form η on a complex manifold X is generated by the relations $x \sim_U y$, where U is an open set on which $\eta = df$ with f holomorphic and x, y are in U and are in the same connected component of a fiber of f.

Theorem 1.4 ([**16**, 4.1]). — *Let* X *be a complete Kähler manifold of bounded geometry,* η *a closed holomorphic 1-form on* X *and* $\mathcal{F}_\eta$ *the associated singular holomorphic foliation. If* $\mathcal{F}_\eta$ *has one compact leaf, all leaves are compact.*

One important principle here, which is an immediate consequence of the volume monotonicity property of analytic subsets of $\mathbf{C}^n$ leading to the definition of Lelong numbers [**13**, 15.1, Prop. 1], is the following uniform boundedness property of submanifolds of finite volume.

Proposition 1.5. — *If* X *is Kähler, complete and of bounded geometry, then for every* $T > 0$ *and* $\epsilon > 0$ *there is* $N(T, \epsilon) \in \mathbf{N}$ *such that every closed (as a subset of* Y*) complex submanifold* $Y \subset X$ *with* $vol(Y) \leq T$ *can be covered by* $N(T, \epsilon)$ *balls of radius* ϵ. *In particular* Y *is compact.*

The proof of Theorem 1.2 then proceeds as follows: let $\alpha \in \mathcal{H}^1_{(2)}(X)$ and η_α be the L^2-holomorphic 1-form with $\alpha = \Re\eta_\alpha$. Let $f : X \to \mathbf{C}$ be holomorphic with $df = \eta_\alpha$; the co-area formula together with the L^2-condition implies that f has a fiber of finite volume. This implies by the above fact that $\mathcal{F}_{\eta_\alpha}$ has a compact leaf and by Theorem 1.4 that all leaves are compact, so that one can apply Stein factorization. The final point consists in showing that $\mathcal{F}_{\eta_\alpha}$ does not depend on the particular choice of α; this follows from a tricky argument in L^2-Hodge theory, using the boundedness of η_α ([**22**] or [**1**, lemma 4.16]) which gives that $\eta_\alpha \wedge \eta_\beta = 0$ for any choice $\alpha, \beta \in \mathcal{H}^1_{(2)}(X)$, and hence $\mathcal{F}_{\eta_\alpha} = \mathcal{F}_{\eta_\beta}$.