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(1039) *Invariant percolation and measured theory of nonamenable groups*

Cyril HOUDAYER

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
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INVARIANT PERCOLATION AND MEASURED THEORY OF NONAMENABLE GROUPS
[after Gaboriau-Lyons, Ioana, Epstein]

by **Cyril HOUDAYER**

1. INTRODUCTION

The notion of amenability was introduced in 1929 by J. von Neumann [**48**] in order to explain the Banach-Tarski paradox. A countable discrete group Γ is *amenable* if there exists a left-invariant mean $\varphi : \ell^\infty(\Gamma) \to \mathbf{C}$. The class of amenable groups is stable under subgroups, direct limits, quotients and the free group $\mathbf{F}_2$ on two generators is not amenable. Knowing whether or not the class of amenable groups coincides with the class of groups without a nonabelian free subgroup became known as von Neumann's problem. It was solved in the negative by Ol'shanskii [**50**]. Adyan [**1**] proved that the free Burnside groups $B(m, n)$ with m generators, of exponent n ($n \geq 665$ and odd) are nonamenable. Ol'shanskii and Sapir [**51**] also constructed examples of finitely presented nonamenable groups without a nonabelian free subgroup.

Two free ergodic probability measure-preserving (pmp) actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ of countable discrete groups on nonatomic standard probability spaces are *orbit equivalent* (OE) if they induce the same orbit equivalence relation, that is, if there exists a pmp Borel isomorphism $\Delta : (X, \mu) \to (Y, \nu)$ such that $\Delta(\Gamma x) = \Lambda\Delta(x)$, for μ-almost every $x \in X$. Despite the fact that the group $\mathbf{Z}$ admits uncountably many non-conjugate free ergodic pmp actions, Dye [**13, 14**] proved the surprising result that any two free ergodic pmp actions of $\mathbf{Z}$ are orbit equivalent. Moreover, Ornstein and Weiss [**52**] (see also [**11**]) proved that any free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$ of any infinite amenable group is always orbit equivalent to a free ergodic pmp $\mathbf{Z}$-action on (X, μ). On the other hand, results of [**62, 12, 26**] imply that any nonamenable group has at least two non-OE free ergodic pmp actions. These results lead to a satisfying

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characterization of amenability: an infinite countable discrete group Γ is amenable if and only if Γ admits exactly one free ergodic pmp action up to OE.

Measurable-group-theoretic solution to von Neumann's problem

The first result we discuss in this paper is a positive answer to von Neumann's problem in the framework of measured group theory, due to Gaboriau and Lyons **[22]**. Measured group theory is the study of countable discrete groups Γ through their pmp actions $\Gamma \curvearrowright (X, \mu)$. We refer to **[21]** for a recent survey on this topic.

To any free pmp action $\Gamma \curvearrowright (X, \mu)$, one can associate the *orbit equivalence relation* $\mathcal{R}(\Gamma \curvearrowright X) \subset X \times X$ defined by

$$(x, y) \in \mathcal{R}(\Gamma \curvearrowright X) \iff \exists g \in \Gamma, y = gx.$$

For countable discrete groups Γ and Λ, we say that Λ is a *measurable subgroup* of Γ and set $\Lambda <_{\mathrm{ME}} \Gamma$ if there exist two free ergodic pmp actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (X, \mu)$ such that $\mathcal{R}(\Lambda \curvearrowright X) \subset \mathcal{R}(\Gamma \curvearrowright X)$. Denote by Leb the Lebesgue measure on the interval $[0, 1]$ and let $\Gamma \curvearrowright ([0, 1], \mathrm{Leb})^{\Gamma}$ be the Bernoulli shift. Gaboriau and Lyons **[22]** obtained the following remarkable result.

THEOREM. — *Let Γ be any nonamenable countable discrete group. Then there exists a free ergodic pmp action $\mathbf{F}_2 \curvearrowright ([0, 1], \mathrm{Leb})^{\Gamma}$ such that*

$$\mathcal{R}(\mathbf{F}_2 \curvearrowright [0, 1]^{\Gamma}) \subset \mathcal{R}(\Gamma \curvearrowright [0, 1]^{\Gamma}).$$

In particular, we get that $\mathbf{F}_2 <_{\mathrm{ME}} \Gamma$. This theorem has important consequences in the theory of group von Neumann algebras.

COROLLARY. — *Let Γ, H be countable discrete groups such that Γ is nonamenable and H is infinite. Then the von Neumann algebra $L(H \wr \Gamma)$ of the wreath product group $H \wr \Gamma := (\bigoplus_{\Gamma} H) \rtimes \Gamma$ contains a copy of the von Neumann algebra $L(\mathbf{F}_2)$ of the free group.*

The proof of Gaboriau and Lyons' result goes in two steps that we explain below. We refer to Section 2 for background material on pmp equivalence relations.

The **first step** consists in finding a subequivalence relation $\mathcal{R} \subset \mathcal{R}(\Gamma \curvearrowright [0, 1]^{\Gamma})$ such that $\mathcal{R}$ is ergodic treeable and non-hyperfinite. This is a difficult problem in general. By Zimmer's result **[68**, Proposition 9.3.2], it is known that $\mathcal{R}(\Gamma \curvearrowright [0, 1]^{\Gamma})$ contains an ergodic hyperfinite subequivalence relation. When Γ is finitely generated, another way to obtain subequivalence relations of $\mathcal{R}(\Gamma \curvearrowright [0, 1]^{\Gamma})$ is by considering invariant *percolation* processes on the Cayley graphs of Γ (see Section 3). This beautiful idea is due to Gaboriau **[20]**. Gaboriau and Lyons exploit this idea and give two different proofs of the first step, one using random forests, the other using Bernoulli percolation. They also suggest at the end of their article that the *free minimal spanning forest* [**41**] could serve as the desired treeable non-hyperfinite subequivalence relation $\mathcal{R}$. It is this approach that we will present in this paper. Sections 2 through 7 are entirely devoted to giving a self-contained proof of this first step. The proof is a combination of ideas and techniques involving probability, ergodic theory, geometric group theory and von Neumann algebras theory.

In the **second step**, one uses Gaboriau's theory of cost [**18**] (see also [**35**]). An ergodic treeable non-hyperfinite equivalence relation has cost greater than 1 by [**18**, Théorème IV.1]. From the first step, one can then construct an ergodic treeable subequivalence relation $\mathcal{R} \subset \mathcal{R}(\Gamma \curvearrowright [0,1]^\Gamma)$ with cost ≥ 2. Finally, one applies Hjorth's result [**27**] in order to get a subequivalence relation of $\mathcal{R}(\Gamma \curvearrowright [0,1]^\Gamma)$ induced by a free ergodic pmp action of $\mathbf{F}_2$.

Orbit equivalence theory of nonamenable groups

As mentioned before, any nonamenable group admits at least two non-OE free ergodic pmp actions [**12, 26, 62**]. Over the last few years, the following classes of nonamenable groups have been shown to admit uncountably many non-OE free ergodic pmp actions: property (T) groups (Hjorth [**26**]); nonabelian free groups (Gaboriau and Popa [**23**]); weakly rigid groups[1] (Popa [**56**]); nonamenable products of infinite groups (Popa [**60**], see also [**45, 28**]); mapping class groups (Kida [**37**]). We refer to [**5, 24, 68**] for earlier results on this topic.

In his breakthrough paper [**30**], Ioana proved that every nonamenable group Γ that contains $\mathbf{F}_2$ as a subgroup admits uncountably many non-OE free ergodic pmp actions. As we will see in Section 9, Ioana's proof goes in two steps that we outline. Regard $\mathbf{F}_2 < \mathrm{SL}_2(\mathbf{Z})$ as a finite index subgroup and let $\mathbf{F}_2$ act on $\mathbf{Z}^2$ by matrix multiplication. By results of Kazhdan-Margulis [**33, 43**], the pair $(\mathbf{Z}^2 \rtimes \mathbf{F}_2, \mathbf{Z}^2)$ has the relative property (T). Write $\alpha : \mathbf{F}_2 \curvearrowright (\mathbf{T}^2, \lambda^2)$ for the corresponding pmp action. The first step (see Theorem 9.1) shows that in every uncountable set of mutually OE actions of Γ whose restrictions to $\mathbf{F}_2$ admit α as a quotient, we can find two actions whose restrictions to $\mathbf{F}_2$ are conjugate. The proof is based on a separability argument which uses in a crucial way the fact that the action $\alpha : \mathbf{F}_2 \curvearrowright \mathbf{T}^2$ is *rigid* in the sense of Popa [**55**]. Note that the action α was already successfully used by Gaboriau and Popa [**23**] in order to show that the free groups $\mathbf{F}_n$ have a continuum of non-OE actions. The second step consists in using the co-induction technique (see Section 8) in order to construct uncountably many actions of Γ whose restrictions to $\mathbf{F}_2$ are non-conjugate. Altogether, one obtains uncountably many non-OE actions of Γ.

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[1] A countable Γ is weakly rigid in the sense of Popa if it admits an infinite normal subgroup $\Lambda < \Gamma$ such that the pair (Γ, Λ) has the relative property (T).

Gaboriau and Lyons' result opened up the possibility that the condition "Γ contains $\mathbf{F}_2$" in Ioana's theorem could be replaced by the natural condition "Γ is nonamenable". In order to do so, one had to generalize the second step of Ioana's proof, that is, one needed a more general co-induction technology for group/measurable subgroup rather than group/subgroup. Epstein [**15**] obtained such a construction (see Section 8). Since the first step of Ioana's proof remains unchanged for Γ containing $\mathbf{F}_2$ as a measurable subgroup, Epstein [**15**] obtained the following result.

THEOREM. — *Every nonamenable group* Γ *admits uncountably many non-OE free ergodic pmp actions.*

Since then, this result has been generalized in two ways. First, recall that any free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$ gives rise to a finite von Neumann algebra $L^\infty(X) \rtimes \Gamma$ via the *group measure space construction* of Murray and von Neumann (see Section 6). Two free ergodic pmp actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are W*-*equivalent* if the von Neumann algebras $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$ are $*$-isomorphic. Since the group measure space construction only depends on the orbit structure of the action [**63**] (see also [**17**]), it follows that orbit equivalence implies W*-equivalence. Using Popa's concept of *rigid inclusion* of von Neumann algebras [**55**], Ioana [**30**] strengthened the previous result by showing that any nonamenable group Γ admits a continuum of W*-inequivalent free ergodic pmp actions. Next, given any nonamenable group Γ, denote by $A_0(\Gamma, X, \mu)$ the standard Borel space of all free mixing pmp actions of Γ on (X, μ) (see [**34**]). On the space $A_0(\Gamma, X, \mu)$, consider the Borel equivalence relation OE defined by $(a, b) \in$ OE if and only if the actions a and b are orbit equivalent. Epstein, Ioana, Kechris and Tsankov [**31**] proved that OE on the space $A_0(\Gamma, X, \mu)$ cannot be classified by countable structures.

We point out that both Ioana's theorem and Epstein's theorem rely on a separability argument and therefore only provide the *existence* of a continuum of non-OE actions for Γ. What about concrete examples of a continuum of non-OE actions for a given nonamenable group Γ? Important progress has been made over the recent years. The classes of nonamenable groups for which a concrete uncountable family of non-OE actions is known are the following: non-abelian free groups (Ioana [**29**]); weakly rigid groups (Popa [**56**]); nonamenable products of infinite groups (Popa [**60**]); mapping class groups (Kida [**37**]). We also refer to Popa and Vaes [**61**] for further results regarding this question.

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