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(1040) *A Global Torelli theorem for hyperkähler manifolds*

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A GLOBAL TORELLI THEOREM FOR HYPERKÄHLER MANIFOLDS [after M. Verbitsky]

by **Daniel HUYBRECHTS**

Compact hyperkähler manifolds are higher-dimensional generalizations of K3 surfaces. The classical Global Torelli theorem for K3 surfaces, however, does not hold in higher dimensions. More precisely, a compact hyperkähler manifold is in general not determined by its natural weight-two Hodge structure. The text gives an account of a recent theorem of M. Verbitsky, which can be regarded as a weaker version of the Global Torelli theorem phrased in terms of the injectivity of the period map on the connected components of the moduli space of marked manifolds.

1. INTRODUCTION

The Global Torelli theorem is said to hold for a particular class of compact complex algebraic or Kähler manifolds if any two manifolds of the given type can be distinguished by their integral Hodge structures.

The most prominent examples for which a Global Torelli theorem has been proved classically include complex tori and complex curves. Two complex tori $T = \mathbb{C}^n/\Gamma$ and $T' = \mathbb{C}^n/\Gamma'$ are biholomorphic complex manifolds if and only if there exists an isomorphism of weight-one Hodge structures $H^1(T,\mathbb{Z}) \cong H^1(T',\mathbb{Z})$. Similarly, two smooth compact complex curves C and C' are isomorphic if and only if there exists an isomorphism of weight-one Hodge structures $H^1(C,\mathbb{Z}) \cong H^1(C',\mathbb{Z})$ that in addition respects the intersection pairing.

Here, we are interested in higher-dimensional analogues of the following Global Torelli theorem for K3 surfaces.

• *Two complex K3 surfaces S and S' are isomorphic if and only if there exists an isomorphism of Hodge structures $H^2(S,\mathbb{Z}) \cong H^2(S',\mathbb{Z})$ respecting the intersection pairing.*

The result is originally due to Pjateckiĭ-Šapiro and Šafarevič in the algebraic case and to Burns and Rapoport for K3 surfaces of Kähler type (but as Siu proved later, every K3 surface is in fact Kähler), see [**4, 2**] for details and references.

Recall that for complex tori T and T', any Hodge isomorphism $H^1(T,\mathbb{Z}) \cong H^1(T',\mathbb{Z})$ is induced by an isomorphism $T \cong T'$. Also, for complex curves any Hodge isometry can be lifted up to sign. A similar stronger form of the Global Torelli theorem holds for generic K3 surfaces.

• *For any Hodge isometry $\varphi : H^2(S,\mathbb{Z}) \xrightarrow{\sim} H^2(S',\mathbb{Z})$ between two generic (!) K3 surfaces, there exists an isomorphism $g : S \xrightarrow{\sim} S'$ with $\varphi = \pm g_*$.*

1.1. Is there a Global Torelli for hyperkähler manifolds?

Compact hyperkähler manifolds are the natural higher-dimensional generalizations of K3 surfaces and it would be most interesting to establish some version of the Global Torelli theorem for this important class of Ricci-flat manifolds. In this context, the second cohomology $H^2(X,\mathbb{Z})$ is the most relevant part of cohomology and not the much larger middle cohomology which one would usually consider for arbitrary compact Kähler manifolds. As we will recall in Section 2, the second cohomology of a compact hyperkähler manifold comes with a natural quadratic form, the Beauville–Bogomolov form, and its canonical weight-two Hodge structure is of a particularly simple type.

So, is a compact hyperkähler manifold X determined up to isomorphism by its weight-two Hodge structure $H^2(X,\mathbb{Z})$ endowed with the Beauville–Bogomolov form? More precisely, are two compact hyperkähler manifolds X and X' isomorphic if $H^2(X,\mathbb{Z})$ and $H^2(X',\mathbb{Z})$ are Hodge isometric, i.e. if there exists an isomorphism of weight-two Hodge structures $H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ that is compatible with the Beauville–Bogomolov forms on both sides? Unfortunately, as was discovered very early on, a Global Torelli theorem for compact hyperkähler manifolds cannot hold true literally.

The first counterexample was produced by Debarre in [**8**]:

• *There exist non-isomorphic compact hyperkähler manifolds X and X' with isometric weight-two Hodge structures.*

See also [**25**, Ex. 7.2] for examples with X and X' projective (and in fact isomorphic to certain Hilbert schemes of points on projective K3 surfaces).

In Debarre's example X and X' are bimeromorphic and for quite some time it was hoped that $H^2(X,\mathbb{Z})$ would at least determine the bimeromorphic type of X. As two bimeromorphic K3 surfaces are always isomorphic, a result of this type would still qualify as a true generalization of the Global Torelli theorem for K3 surfaces. This hope was shattered by Namikawa's example in [**21**]:

• *There exist compact hyperkähler manifolds X and X' (projective and of dimension four) with isometric Hodge structures $H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ but without X and X' being bimeromorphic (birational).*

Nevertheless, at least for the time being the second cohomology of a compact hyperkähler manifold is still believed to encode most of the geometric information on the manifold. Possibly other parts of the cohomology might have to be added, but no convincing general version of a conjectural Global Torelli theorem using more than the second cohomology has been put forward so far.

At the moment it seems unclear what the existence of a Hodge isometry $H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ between two compact hyperkähler manifolds could mean concretely for the relation between the geometry of X and X' (but see Corollary 6.5 for special examples). However, rephrasing the classical Torelli theorem for K3 surfaces in terms of moduli spaces suggests a result that was eventually proved by Verbitsky in [**24**].

1.2. Global Torelli via moduli spaces

The following rather vague discussion is meant to motivate the main result of [**24**] to be stated in the next section. The missing details and precise definitions will be given later.

1.2.1. — We start by rephrasing the Global Torelli theorem for K3 surfaces using the moduli space $\mathfrak{M}$ of marked K3 surfaces and the period map $\mathscr{P} : \mathfrak{M} \to \mathbb{P}(\Lambda \otimes \mathbb{C})$. A marked K3 surface (S,ϕ) consists of a K3 surface S and an isomorphism of lattices $\phi : H^2(S,\mathbb{Z}) \xrightarrow{\sim} \Lambda$, where $\Lambda := 2(-E_8) + 3U$ is the unique even unimodular lattice of signature $(3,19)$. Two marked K3 surfaces (S,ϕ) and (S',ϕ') are isomorphic if there exists an isomorphism (i.e. a biholomorphic map) $g : S \xrightarrow{\sim} S'$ with $\phi \circ g^* = \phi'$. Then by definition $\mathfrak{M} = \{(S,\phi)\}/_{\cong}$.

The Global Torelli theorem for K3 surfaces is equivalent to the following statement.

• *The moduli space $\mathfrak{M}$ has two connected components interchanged by $(S,\phi) \mapsto (S,-\phi)$ and the period map*

$$\mathscr{P} : \mathfrak{M} \to D_\Lambda := \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid x^2 = 0,\ (x.\bar{x}) > 0\},\ (S,\phi) \mapsto [\phi(H^{2,0}(S))]$$

is generically injective on each of the two components.

Remark 1.1. — Injectivity really only holds generically, i.e. for (S,ϕ) in the complement of a countable union of hypersurfaces (cf. Remark 3.2). This is related to the aforementioned stronger form of the Global Torelli theorem being valid only for generic K3 surfaces.

Let us now consider the natural action

$$\mathrm{O}(\Lambda)\times\mathfrak{M}\to\mathfrak{M},\ (\varphi,(S,\phi))\mapsto(S,\varphi\circ\phi).$$

For any $(S,\phi)\in\mathfrak{M}^o$ in a connected component $\mathfrak{M}^o$ of $\mathfrak{M}$ the subgroup of $\mathrm{O}(\Lambda)$ that fixes $\mathfrak{M}^o$ is $\phi\circ\mathrm{Mon}(X)\circ\phi^{-1}$, where the monodromy group $\mathrm{Mon}(S)\subset\mathrm{O}(H^2(S,\mathbb{Z}))$ is by definition generated by all monodromies $\pi_1(B,t)\to\mathrm{O}(H^2(S,\mathbb{Z}))$ induced by arbitrary smooth proper families $\mathcal{X}\to B$ with $\mathcal{X}_t=S$.

The transformation $-\mathrm{id}\in\mathrm{O}(\Lambda)$ induces the involution $(S,\phi)\mapsto(S,-\phi)$ that interchanges the two connected components and, as it turns out, there is essentially no other $\varphi\in\mathrm{O}(\Lambda)$ with this property. This becomes part of the following reformulation of the Global Torelli theorem for K3 surfaces:

- *Each connected component $\mathfrak{M}^o\subset\mathfrak{M}$ maps generically injectively into D_Λ and for any K3 surface S one has $\mathrm{O}(H^2(S,\mathbb{Z}))/\mathrm{Mon}(S)=\{\pm1\}$.*

In order to show that this version implies the one above, one also needs the rather easy fact that any two K3 surfaces S and S' are deformation equivalent, i.e. that there exist a smooth proper family $\mathcal{X}\to B$ over a connected base and points $t,t'\in B$ such that $S\cong\mathcal{X}_t$ and $S'\cong\mathcal{X}_{t'}$. In particular, all K3 surfaces are realized by complex structures on the same differentiable manifold.

1.2.2. — Let us try to generalize the above discussion to higher dimensions. Restricting to compact hyperkähler manifolds X of a fixed deformation class, the isomorphism type, say Λ, of the lattice realized by the Beauville–Bogomolov form on $H^2(X,\mathbb{Z})$ is unique, cf. Section 2. So the moduli space $\mathfrak{M}_\Lambda$ of Λ-marked compact hyperkähler manifolds of fixed deformation type (the latter condition is not reflected by the notation) can be introduced, cf. Section 4.2 for details.

For the purpose of motivation let us consider the following two statements. Both are false (!) in general, but the important point here is that they are equivalent and that the first half of the second one turns out to be true.

- (Global Torelli, standard form) *Any Hodge isometry $H^2(X,\mathbb{Z})\cong H^2(X',\mathbb{Z})$ between generic X and X' can be lifted up to sign to an isomorphism $X\cong X'$.*

- (Global Torelli, moduli version) i) *On each connected component $\mathfrak{M}^o_\Lambda\subset\mathfrak{M}_\Lambda$ the period map $\mathcal{P}:\mathfrak{M}^o_\Lambda\to D_\Lambda$ is generically injective.* ii) *For any hyperkähler manifold X parametrized by $\mathfrak{M}_\Lambda$ one has $\mathrm{O}(H^2(X,\mathbb{Z}))/\mathrm{Mon}(X)=\{\pm1\}$.*

Remark 1.2. — In both statements, generic is meant in the sense of Remark 1.1. The standard form would then indeed imply that arbitrary Hodge isometric X and X' are bimeromorphic. For details on the passage from generic to arbitrary hyperkähler manifolds and thus to the bimeromorphic version of the Global Torelli theorem, see Section 6.1.