352

ASTÉRISQUE

2013

SÉMINAIRE BOURBAKI
VOLUME 2011/2012
EXPOSÉS 1043-1058

(1047) *The Ribe Programme*

Keith BALL

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Séminaire BOURBAKI Janvier 2012
64$^{\text{e}}$ année, 2011-2012, n$^{\text{o}}$ 1047, p. 147 à 159

THE RIBE PROGRAMME

by **Keith BALL**

INTRODUCTION

In 1976 Ribe [**23**] proved that uniformly homeomorphic Banach spaces have uniformly *linearly* isomorphic finite-dimensional subspaces: that if there is a uniformly continuous homeomorphism between two Banach spaces, with uniformly continuous inverse, then every finite-dimensional subspace of one space is linearly isomorphic to a subspace of the other, with an implied constant of isomorphism independent of the dimension. This remarkable rigidity theorem guarantees that finite-dimensional properties are determined up to isomorphism by the metric structure of the space. Thus in principle, any property of a space that depends upon finite collections of points has an equivalent formulation which makes no reference to the linear structure of the space and involves only the distances between points.

In view of this, Bourgain [**6**] proposed an ambitious programme which came to be known as "The Ribe Programme": to find explicit metric descriptions of the most important invariants of normed linear spaces. He himself kick-started the programme by characterising superreflexivity. A Banach space X is called *superreflexive* if every space whose finite-dimensional spaces embed uniformly well into X, must automatically be reflexive. Bourgain showed this holds precisely if the space does not contain copies of arbitrarily large binary trees. Needless to say the general aim of the Ribe programme is not merely to find metric equivalents of linear properties but to transfer the subtle and well-developed theory of normed spaces to the non-linear setting, and in this broader sense the programme had been anticipated in a prescient paper of Johnson and Lindenstrauss, [**13**].

Within a decade or two the Ribe programme acquired an importance that would have been hard to predict at the outset, as the insights provided by the programme became powerful tools in the theory of algorithms and more recently in geometric group theory. Data sets often come equipped with a metric structure but rarely a linear one. For a number of central algorithmic problems the most effective procedure is to embed the data-set into a familiar linear geometry, without distorting it too much, and then use the linear structure to analyse the embedded copy of the data. Plainly, data that fail to possess a metric property that holds in the linear geometry cannot be embedded: so it is essential to understand what these properties are and where they appear. The study of groups as metric spaces and in particular their embeddability into simpler geometric structures has become a subject of intense interest in the last 5-10 years because of its connection with several famous problems such as the Novikov conjecture.

In this article I will describe what is currently by far the most successful family of achievements within the Ribe programme: the development of metric equivalents of type and cotype. In order to make the article reasonably self-contained I shall discuss the linear invariants as well as the non-linear ones and describe three classical principles (due in various combinations to Kwapień, Maurey and Pisier) that provided the inspiration for the non-linear development and serve as test cases to check that the non-linear invariants are useful. The non-linear theory proceeded in two (or even three) distinct phases that were quite widely separated in time. In the first phase Bourgain and others introduced metric type and initiated the theory of non-linear embedding. Formally the theory of embedding does not form part of the Ribe programme but it goes hand in hand with it. It is hard to imagine that the recent subtle development of the non-linear Dvoretzky Theorem by Mendel and Naor [**19**] would have emerged without something like the Ribe programme. Indeed, it still seems extraordinary that there is a subtle structure theory for objects as diverse as general metric spaces, mirroring the structure theory for normed spaces. In the second phase, or second half of the first phase, the present author introduced the Markov type and cotype properties for the specific purpose of studying Lipschitz extensions. The article had an effect opposite to the one that mathematicians always hope for: it seemed to halt the programme instead of encouraging further development. This was probably because we were unable to prove that the Markov type property held in any space other than Hilbert space. Ten years later this embarrassing open problem was adopted by Naor who solved it in collaboration with Peres, Schramm and Sheffield and who then produced a series of deep articles with Mendel that form the current phase of the development. They introduced a metric form of cotype, showed that for linear spaces it agreed with linear cotype and proved a non-linear analogue of the Maurey-Pisier Theorem. This is the most technically difficult part of the Ribe programme to date.

The story begins with the linear invariants.

1. LINEAR (OR RADEMACHER) TYPE AND COTYPE

The type and cotype invariants were introduced simultaneously by Maurey in the study of factorisation of linear maps and by Hoffmann-Jørgensen for the study of vector-valued central limit theorems. A detailed history of how these ideas grew out of earlier work of James and others can be found in the article of Maurey, [**16**]. A normed space X has type p if there is a constant T so that for every sequence of vectors $x_1, \ldots, x_n$ in X

$$\mathrm{Ave}\| \pm x_1 \pm x_2 \pm \cdots \pm x_n\|^p \leq T^p \sum_1^n \|x_i\|^p$$

where the average is taken over all choices of sign in the vector sum. It has cotype q with constant C if

$$\sum_1^n \|x_i\|^q \leq C^q \mathrm{Ave}\| \pm x_1 \pm x_2 \pm \cdots \pm x_n\|^q$$

for every such sequence. In Hilbert space, both statements hold with equality if $p = 2$ or $q = 2$ and with constants $T = 1$ or $C = 1$. (The identity that results is the parallelogram identity.) Consideration merely of the real line shows that if the respective properties are to hold in any space then necessarily $p \leq 2$ and $q \geq 2$. Using the Khintchine inequality for the L_p norms of sign averages it is straightforward to show that if $1 \leq p \leq 2$ the space L_p has type p and cotype 2, while if $2 \leq q < \infty$, L_q has type 2 and cotype q. Every space has type 1 and (with the obvious convention) cotype ∞. Thus there are spaces other than Hilbert space that possess either the optimal type (type 2) or optimal cotype (cotype 2). For this reason these properties have a special place in the theory. However, a gorgeous result of Kwapień [**14**] from the early days of the theory shows that only Hilbert space can possess both type 2 *and* cotype 2.

2. METRIC TYPE AND A NON-LINEAR ℓ_1 THEOREM

The first metric analogue of type appeared in the work of Enflo [**10**] (actually before the linear version was introduced). He (in effect) asked: for which spaces X is it true that there is a constant T so that for every n and every embedding of the corners of the cube $\{-1, 1\}^n$ into X, the average squared length of the cube's 2^n diagonals is at most nT^2 times the average squared length of the cube's $n2^{n-1}$ edges? Such a space clearly has type 2 (with the same constant) since Enflo's definition allows folded (non-linear) embeddings of the cube, but asks for the same inequality. Enflo observed that Hilbert space *does* have this property: that folding the cube shortens the diagonals (on average) more than the edges. But the question already illustrates some of the difficulties that one encounters in the Ribe programme. It is still unknown whether Enflo's property holds for linear spaces with type 2 and it is clear that we cannot hope to find a metric version of cotype just by allowing folded cubes, since we can fold the diagonals to nothing while keeping the edges of fixed length.

Following Bourgain's enunciation of the Ribe programme, Bourgain, Milman and Wolfson [**8**] wrote an article in which they chose a modification of Enflo's property as their definition of metric type (for $1 < p \leq 2$) and showed that a space with linear type p has metric type r for all $r < p$. They also proved a metric analogue of Pisier's ℓ_1 theorem (see [**21**]) which states that the finite-dimensional L_1-spaces are the only possible obstruction to type:

Theorem 2.1 (Pisier). — *If a normed space X fails to have type p for every $p > 1$ (the space has* no *non-trivial type) then there is a constant C so that for every n, X has a subspace Y which is C-isomorphic to the n-dimensional L_1-space, ℓ_1^n: in other words there is a linear isomorphism $T : Y \to \ell_1^n$ with $\|T\| \, \|T^{-1}\| \leq C$.*

The theorem of [**8**] guarantees that a *metric* space which has no non-trivial *metric* type contains uniformly Lipschitz-equivalent copies of the discrete cube $\{-1, 1\}^n$ equipped with the metric it inherits from ℓ_1^n; the *Hamming cube* as it is usually known.

Theorem 2.2 (Bourgain, Milman, Wolfson). — *If a metric space X fails to have type p for every $p > 1$ (the space has* no *non-trivial metric type) then there is a constant C so that for every n, X has a subset which is C-lipschitz equivalent to the metric space $\{-1, 1\}^n$ with the Hamming metric it inherits from ℓ_1^n.*

As in Pisier's paper, the crucial point is a submultiplicativity property for type constants in terms of the number of vectors. This means that if the space contains a cube of large dimension which looks a bit like a Hamming cube it must contain cubes of smaller dimension which look very like Hamming cubes.

At about the same time, Bourgain [**5**] proved that every n-point metric space can be embedded into Hilbert space with the distances between points of the space being distorted by at most a constant multiple of $\log n$. It had been known since the work of Fritz John [**12**] that two n-dimensional normed spaces are linearly isomorphic with an isomorphism constant at most $\sqrt{n}$. Bourgain's Theorem was clearly inspired by this fact about linear spaces together with a view which emerged at the time that the number of points of a metric space would play a role something like the exponential of the dimension of a normed space. This view was prompted by a fact related to sphere-packing that had by then become a standard tool in the linear theory: the unit ball of an n-dimensional normed space contains an ε-net with no more than $(5/\varepsilon)^n$ elements.