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(1051) *The formation of black holes in general relativity*

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THE FORMATION OF BLACK HOLES IN GENERAL RELATIVITY [after D. Christodoulou]

by **Mihalis DAFERMOS**

INTRODUCTION

Few notions of mathematical physics capture the imagination like that of *black hole*.

The concept was first encountered in explicit solutions of the Einstein vacuum equations:

$$\mathrm{Ric}(g) = 0, \tag{1}$$

specifically the celebrated *Schwarzschild* solution $(\mathcal{M}, g)_{\mathrm{Schw}}$. As was understood already by Lemaitre [**28**], there is a region $\mathcal{B} \subset \mathcal{M}_{\mathrm{Schw}}$ of this spacetime with the property that observers in $\mathcal{B}$ cannot send signals to "far-away" observers. Following J. Wheeler, this region $\mathcal{B}$ is known as the *black hole* (or, in French translation, *le trou noir*).

More generally (and more precisely), one defines the black hole region of an asymptotically flat spacetime $(\mathcal{M}, g)$ as the collection of spacetime points $\mathcal{B} \subset \mathcal{M}$ *not* in the causal past of an ideal conformal boundary at infinity, so-called *future null infinity*, traditionally denoted $\mathcal{I}^+$; in symbols:

$$\mathcal{B} = \mathcal{M} \setminus J^-(\mathcal{I}^+). \tag{2}$$

In the explicit examples of Schwarzschild or Kerr, the existence of a black hole region $\mathcal{B}$ is accompanied by another salient feature: Every timelike or null geodesic $\gamma(s)$ entering the interior of $\mathcal{B}$ is future incomplete. In particular,

$$(\mathcal{M}, g) \text{ is } \textit{future-causally geodesically incomplete}. \tag{3}$$

In the case of Schwarzschild, the curvature grows without bound along *all* incomplete $\gamma(s)$ as the affine parameter s approaches its supremum value. In Kerr, the origin of incompleteness is in some sense even more bizarre; it represents not the breakdown of local regularity but of global causality.

Trapped surfaces and the theorems of Penrose

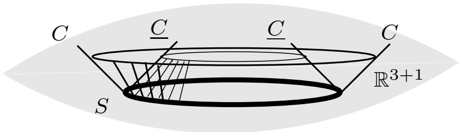
The physical implications of the above two properties (2), (3) are profound. Historically, however, they were very difficult to accept. A common point of view in the early years of the development of general relativity was to bet on an obvious way out of dealing with their consequences:

> *Could it be that the above "black hole"* (2) *and incompleteness properties* (3) *are pathologies, due to the high degree of symmetry of explicit solutions?*

One of the great successes of the global geometrical approach first pioneered by Penrose in the 1960s, was that it provided a definitive answer to the above question *in the negative.*

The key to this answer is provided by a fundamental notion introduced by Penrose [**34**], that of a *closed trapped surface.*

To motivate this notion, let us begin for sake of comparison with a standard 2-sphere S in Minkowski space $\mathbb{R}^{3+1}$ of radius 1.



If we consider the future of S, denoted $J^+(S)$, its boundary in $\mathbb{R}^{3+1}$ has two connected components, the two null cones C and $\underline{C}$ depicted. Considering the second fundamental form χ, $\underline{\chi}$ of S viewed as a hypersurface in each of the above null cones, respectively, we have that

$$\mathrm{tr}\underline{\chi} = -2 < 0, \qquad \mathrm{tr}\chi = 2 > 0.$$

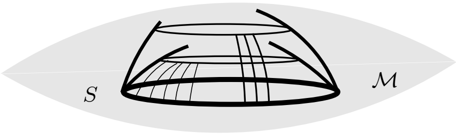
We call $\mathrm{tr}\underline{\chi}$, $\mathrm{tr}\chi$ the *future expansions* because they measure the change in the area element of the flow of S along the null generators of the respective cones.

Given now a general 4-dimensional time-oriented Lorentzian manifold $(\mathcal{M}, g)$, and a closed two surface S, we may again define the two second fundamental forms $\underline{\chi}$ and χ corresponding to viewing S as a hypersurface in each of the two connected components of the boundary of $J^+(S)$ intersected with a tubular neighborhood of S in $\mathcal{M}$. (Again, these are null hypersurfaces generated by the two sets of null geodesics orthogonal to S.)

We say that S is *trapped* if **both** its future expansions are negative:

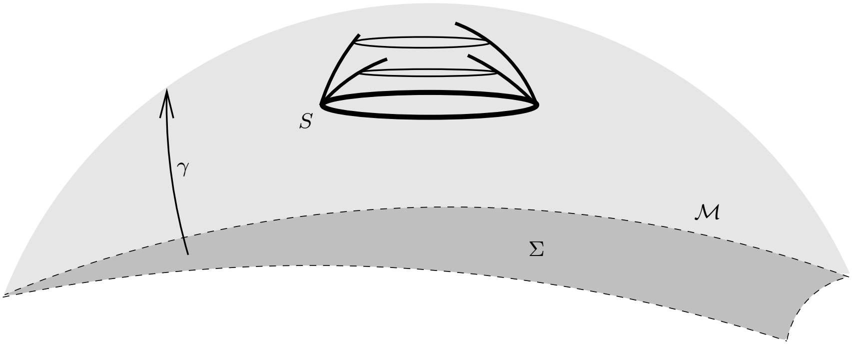
$$\mathrm{tr}\underline{\chi} < 0, \qquad \mathrm{tr}\chi < 0.$$

This is depicted here:



Penrose's celebrated *incompleteness theorem* then states:

THEOREM 0.1 (Penrose, 1965). — *Let* $(\mathcal{M}, g)$ *be globally hyperbolic with a non-compact Cauchy hypersurface, and let* $\mathcal{M}$ *satisfy* $\mathrm{Ric}(V, V) \geq 0$ *for all null* V*. It follows that if* $\mathcal{M}$ *contains a closed trapped surface, then it is future causally geodesically complete.*



Future causal geodesic incompleteness means that there exists an inextendible future-directed timelike or null geodesic γ whose maximum affine parameter is bounded above.

Note of course that when the vacuum equations (1) are assumed, the condition $\mathrm{Ric}(V, V) \geq 0$ is trivially satisfied.

The Einstein vacuum equations have a well-posed Cauchy problem (see Section 1.1.4). The statement of the theorem is such that it can be immediately applied to the maximal Cauchy development of asymptotically flat vacuum initial data (*if it is assumed that the spacetime contains a trapped surface* S), since the assumption of global hyperbolicity holds (by fiat!) for Cauchy developments (see Section 1.1.5). Let us note that by Cauchy stability, the existence of a closed trapped surface is now manifestly a stable property under perturbation of initial data. We obtain in particular the following:

COROLLARY 0.2. — *Let* $(\Sigma, \bar{g}, K)$ *be a sufficiently small perturbation of Schwarzschild data for the Einstein vacuum equations* (1)*. Then the maximal Cauchy development* $(\mathcal{M}, g)$ *contains a closed trapped surface* S *and is geodesically incomplete.*

Concerning the black hole property, again one can state a very general result, assuming that one can define an appropriate notion of conformal boundary "at infinity", denoted $\mathscr{I}^+$, representing future null infinity. Without going into the details of such a definition (see Section 1.3), let us state:

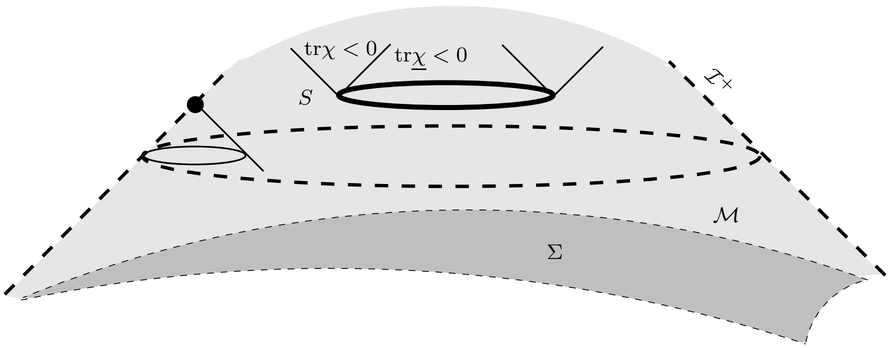
THEOREM 0.3 (See **[21, 42, 11]**). — *Under the assumptions of the previous theorem, if* Σ *is asymptotically flat and* $\mathscr{I}^+$ *is a suitable conformal boundary representing future null infinity, then*

$$S \cap J^-(\mathscr{I}^+) = \varnothing,$$

in particular

$$\mathcal{M} \setminus J^-(\mathscr{I}^+) \neq \varnothing. \tag{4}$$

Because this result makes reference to the causal structure, in depicting this, it is more appropriate to draw the light cones as if they are Minkowskian, so causal relations can be readily understood. Now, however, the area element is not to be inferred by the "size" of the cross-sections, and the signs of the expansions must be labelled:



Our depictions in what follows will typically be of this form.

Similarly to Corollary 0.2, it follows that for sufficiently small perturbations of Schwarzschild data, the resulting Cauchy development $(\mathcal{M}, g)$ will still contain a (nontrivial) black hole region $\mathcal{B}$ (where the latter is defined simply in the sense of (2); see however Section 14!).

The main result: the dynamic formation of trapped surfaces

General as the above results may be, they do not shed light on whether black holes actually form *in nature*. The reason is that the assumption of the existence of a trapped surface is already a very strong assumption about the geometry of spacetime, one that—though stable—a priori may have nothing to do with properties of spacetimes that arise in physically interesting systems. Indeed, the only way previously known to