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CATEGORIFICATION OF LIE ALGEBRAS
[after Rouquier, Khovanov-Lauda, ...]

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INTRODUCTION

Categorification is the process of finding hidden higher level structure. To categorify a natural number, we look for a vector space whose dimension is that number. For example, the passage from Betti numbers to homology groups was an important advance in algebraic topology.

To categorify a vector space V, we look for a category $\mathcal{C}$ whose Grothendieck group is that vector space, $K(\mathcal{C}) = V$. If V carries an action of a Lie algebra $\mathfrak{g}$, then it is natural to look for functors $F_a : \mathcal{C} \to \mathcal{C}$ for each generator a of $\mathfrak{g}$, such that F_a gives the action of a on the Grothendieck group level. In this case, we say that we have categorified the representation V.

There are two general motivations for trying to categorify representations. First, by studying the category $\mathcal{C}$, we hope to learn more about the vector space V. For example, we get a special basis for V coming from classes of indecomposable objects of $\mathcal{C}$. Second, we may use the action of $\mathfrak{g}$ on $\mathcal{C}$ to learn more about $\mathcal{C}$. For example, Chuang-Rouquier used categorification to prove Broué's abelian defect group conjecture for symmetric groups.

Recently, there has been amazing progress towards constructing categorifications of representations of semisimple (or more generally Kac-Moody) Lie algebras. In this report, we aim to give an introduction to this theory. We start with the categorification of $\mathfrak{sl}_2$ and its representations. We explain the naive definition and then the "true" definition, due to Chuang-Rouquier [CR]. We also explain how this definition leads to interesting equivalences of categories. We then address general Kac-Moody Lie algebras, reaching the definition of the Khovanov-Lauda-Rouquier 2-category [R2, KL3]. We explain the relationship to Lusztig's categories of perverse sheaves, due to Varagnolo-Vasserot [VV] and Rouquier [R3]. We close by discussing three fundamental examples of categorical representations: modular representation theory of symmetric groups (due to Lascoux-Leclerc-Thibon [LLT], Grojnowski [Gr], and Chuang-Rouquier [CR]), cyclotomic quotients of KLR algebras (due to Kang-Kashiwara [KK] and Webster [W1]), and quantized quiver varieties (due to Zheng [Z] and Rouquier [R3]).

In order to keep the exposition readable, we have made a number of simplifications and glossed over many details. In particular, we only address simply-laced Kac-Moody Lie algebras (and when it comes to the geometry, only finite-type). We suggest that interested readers consult the literature for more details.

Throughout this paper, we work over $\mathbb{C}$; all vector spaces are $\mathbb{C}$-vector spaces (sometimes they are actually $\mathbb{C}(q)$-vector spaces) and all additive categories are $\mathbb{C}$-linear.

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1. CATEGORIFICATION OF $\mathfrak{sl}_2$ REPRESENTATIONS

1.1. The structure of finite-dimensional representations

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has the basis

$$e = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \ h = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \ f = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right].$$

Consider a finite-dimensional representation V of $\mathfrak{sl}_2$. A basic theorem of representation theory states that h acts semisimply on V with integer eigenvalues. Thus we may write $V = \oplus_{r \in \mathbb{Z}} V_r$ as the direct sum of the eigenspaces for h. Moreover the commutation relations between the generators e, f, h imply the following.

(1) For each r, e restricts to a linear map $e : V_r \to V_{r+2}$.

(2) Similarly, f restricts to a linear map $f : V_r \to V_{r-2}$.

(3) These restrictions obey the commutation relation

$$ef - fe|_{V_r} = rI_{V_r}. \tag{1}$$

Conversely, a graded vector space $V = \oplus V_r$, along with raising and lowering operators e, f as above, defines a representation of $\mathfrak{sl}_2$ if these operators satisfy the relation (1).

The following example will be very instructive.

Example 1.1. — Let X be a finite set of size n. Let $V = \mathbb{C}^{P(X)}$ be a vector space whose basis consists of the subsets of X. For $r = -n, -n+2, \dots, n$, define V_r to be the span of subsets of size k, where $r = 2k - n$.

Define linear maps $e : V_r \to V_{r+2}$, $f : V_r \to V_{r-2}$ by the formulas

$$e(S) = \sum_{T \supset S, |T| = |S|+1} T, \qquad f(S) = \sum_{T \subset S, |T| = |S|-1} T \tag{2}$$

It is easy to check that $(ef - fe)(S) = (2k - n)S$, if S has size k. (The basic reason is that there are $n - k$ ways to add something to S and k ways to take something away from S.)

Thus this defines a representation of $\mathfrak{sl}_2$. In fact, this representation is isomorphic to an n-fold tensor product $(\mathbb{C}^2)^{\otimes n}$ of the standard representation of $\mathfrak{sl}_2$.

We will also need the concept of a representation of the quantum group $U_q\mathfrak{sl}_2$, though we will neither need nor give an explicit definition of $U_q\mathfrak{sl}_2$.

For each integer r, let

$$[r] := \frac{q^r - q^{-r}}{q - q^{-1}} = q^{r-1} + q^{r-3} + \cdots + q^{-r+1}$$

denote the quantum integer (the second expression is only valid if $r \geqslant 0$).

A representation of $U_q\mathfrak{sl}_2$ is a graded $\mathbb{C}(q)$ vector space $V = \oplus V_r$ along with raising $e : V_r \to V_{r+2}$ and lowering $f : V_r \to V_{r-2}$ operators such that $ef - fe|_{V_r} = [r] I_{V_r}$.

1.2. Naive categorical action

Once we think of an $\mathfrak{sl}_2$ representation in terms of a sequence of vector spaces together with raising and lowering operators, we are led to the notion of an action of $\mathfrak{sl}_2$ on a category.

Definition 1.2. — *A naive categorical $\mathfrak{sl}_2$ action consists of a sequence D_r of additive categories along with additive functors $E : D_r \to D_{r+2}$, $F : D_r \to D_{r-2}$, for each r, such that there exist isomorphisms of functors*

$$EF|_{D_r} \cong FE|_{D_r} \oplus I_{D_r}^{\oplus r}, \quad \text{if } r \geqslant 0 \tag{3}$$

$$FE|_{D_r} \cong EF|_{D_r} \oplus I_{D_r}^{\oplus r}, \quad \text{if } r \leqslant 0 \tag{4}$$

Suppose that the categories D_r carry a naive categorical $\mathfrak{sl}_2$ action. Then we can construct a usual $\mathfrak{sl}_2$ representation as follows. We set $V_r = K(D_r)$, the complexified split Grothendieck group. The functors E, F give rise to linear maps $e : V_r \to V_{r+2}$, $f : V_r \to V_{r-2}$ and we can easily see that (3) and (4) give the commutation relation (1). Thus we get a representation of $\mathfrak{sl}_2$ on $V = \oplus V_r$. We say that the categories D_r categorify the representation $V = \oplus V_r$.

It is also useful to consider a graded version of the above definition. A graded additive category is a category $\mathcal{C}$ along with an additive functor $\langle 1\rangle : \mathcal{C} \to \mathcal{C}$. We define a graded naive categorical $\mathfrak{sl}_2$ action as above but with (3), (4) replaced by

$$EF|_{D_r} \cong FE|_{D_r} \oplus I_{D_r}\langle r-1\rangle \oplus \cdots \oplus I_{D_r}\langle -r+1\rangle, \quad \text{if } r \geqslant 0$$
$$FE|_{D_r} \cong EF|_{D_r} \oplus I_{D_r}\langle r-1\rangle \oplus \cdots \oplus I_{D_r}\langle -r+1\rangle, \quad \text{if } r \leqslant 0$$

The Grothendieck groups $K(D_r)$ will then carry an action of $U_q\mathfrak{sl}_2$.

We will now give an example of a naive categorical action which will build on Example 1.1.

In Example 1.1, we studied subsets of a finite set. There is a well-known analogy between subsets of an n-element set and subspaces of an n-dimensional vector space over a finite field $\mathbb{F}_q$, where q is a power of a prime. This analogy suggests that we try to construct a representation of $\mathfrak{sl}_2$ on $\oplus V_r$, where $V_r = \mathbb{C}^{G(k,\mathbb{F}_q^n)}$ is a $\mathbb{C}$-vector space whose basis is $G(k, \mathbb{F}_q^n)$, the set of k-dimensional subspaces of $\mathbb{F}_q^n$ (where $r = 2k - n$ as before). If we define e, f as in (2), then we get a representation of the quantum group $U_{\sqrt{q}}\mathfrak{sl}_2$ (after a slight modification).

The finite set $G(k, \mathbb{F}_q^n)$ is the set of $\mathbb{F}_q$-points of a projective variety, called the Grassmannian. By Grothendieck's fonctions-faisceaux correspondence, we can categorify $\mathbb{C}^{G(k,\mathbb{F}_q^n)}$ using an appropriate category of sheaves on $G(k, \overline{\mathbb{F}}_q^n)$. For simplicity, we switch to characteristic 0 and consider sheaves on $G(k, \mathbb{C}^n)$, the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$.

For each $r = -n, -n+2, \ldots, n$, we let $D_r = D_c^b(G(k, \mathbb{C}^n))$ denote the bounded derived category of constructible sheaves (again here $r = 2k - n$). These are graded categories, where the grading comes from homological shift. With the above motivations, we will define a categorical $\mathfrak{sl}_2$ action using these categories.

For each k, we define the 3-step partial flag variety

$$Fl(k, k+1, \mathbb{C}^n) = \{0 \subset V \subset V' \subset \mathbb{C}^n : \dim V = k, \dim V' = k+1\}$$

$Fl(k, k+1, \mathbb{C}^n)$ serves as a correspondence between $G(k, \mathbb{C}^n)$ and $G(k+1, \mathbb{C}^n)$ and thus it can be used to define functors between categories of sheaves on these varieties. Let $p : Fl(k, k+1, \mathbb{C}^n) \to G(k, \mathbb{C}^n)$ and $q : Fl(k, k+1, \mathbb{C}^n) \to G(k+1, \mathbb{C}^n)$ denote the two projections.

We define

$$E : D_r = D_c^b(G(k, \mathbb{C}^n)) \to D_{r+2} = D_c^b(G(k+1, \mathbb{C}^n))$$
$$\mathcal{A} \mapsto q_*(p^*\mathcal{A})$$
$$F : D_r \to D_{r-2}$$
$$\mathcal{A} \mapsto p_*(q^*\mathcal{A})$$

The above definition of E, F parallels the definition (2).

The following result was proven in an algebraic context (*i.e.*, after applying the Beilinson-Bernstein correspondence) by Bernstein-Frenkel-Khovanov [BFK].

Theorem 1.3. — *This defines a graded naive categorical $\mathfrak{sl}_2$ action.*

The proof of this theorem is relatively straightforward. To illustrate the idea, let us fix $V \in G(k, \mathbb{C}^n)$ and consider $A_1 = \{V' : V \subset V', \dim V' = k+1\}$ and $A_2 = \{V' : V \supset V', \dim V' = k-1\}$; these are the varieties of ways to increase or decrease V. Note that A_1 is a projective space of dimension $n-k-1$ and A_2 is a projective space of dimension $k-1$. Thus $\dim H^*(A_2) - \dim H^*(A_1) = 2k - n$. This observation combined with the decomposition theorem proves the above result.

Remark 1.4. — The Grothendieck group of these categories D_r is actually infinite-dimensional. To cut down to a finite dimensional situation, we can consider the full subcategories $D'_r = P_{Sch}(G(k, \mathbb{C}^n))$ consisting of direct sums of homological shifts of IC-sheaves on Schubert varieties. The subcategories D'_r carry a naive categorical $\mathfrak{sl}_2$ action and by considering dimensions of weight spaces, we can see that they categorify the representation $(\mathbb{C}^2)^{\otimes n}$.

1.3. Categorical $\mathfrak{sl}_2$-action

In the definition of naive categorical $\mathfrak{sl}_2$ action, we only demanded that there exist isomorphisms of functors in (3) and (4). We did not specify the data of these isomorphisms. This is very unnatural from the point of view of category theory. However, it is not immediately obvious how to specify these isomorphisms nor what relations these isomorphisms should satisfy.

In their breakthrough paper, Chuang-Rouquier [CR] solved this problem. First, it is natural to assume that the functors E, F be adjoint (this is a categorification of the fact that e, f are adjoint with respect to the Shapovalov form on any finite-dimensional representation of $\mathfrak{sl}_2$).

Now (assume $r \geqslant 0$), we desire to specify a isomorphism of functors

$$(\phi, \psi_0, \dots, \psi_{r-1}) : EF|_{D_r} \to FE|_{D_r} \oplus I_{D_r}^{\oplus r}$$

so $\phi \in \mathrm{Hom}(EF, FE) \cong \mathrm{Hom}(EE, EE)$ (using the adjunction) and $\psi_s \in \mathrm{Hom}(EF, I) \cong \mathrm{Hom}(E, E)$ (again using the adjunction). Thus it is natural to choose two elements $T \in \mathrm{Hom}(EE, EE)$ and $X \in \mathrm{Hom}(E, E)$ such that ϕ corresponds to T and ψ_s corresponds to X^s for $s = 0, \dots, r-1$.

This leads us to the following definition, essentially due to Chuang-Rouquier [CR].

Definition 1.5. — *A categorical $\mathfrak{sl}_2$ action consists of*

(1) *a sequence D_r of additive categories, with $D_r = 0$ for $r \ll 0$,*

(2) *functors $E : D_r \to D_{r+2}$, $F : D_r \to D_{r-2}$, for each r,*