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GEOMETRIC HYPOELLIPTIC LAPLACIAN AND ORBITAL INTEGRALS [after Bismut, Lebeau and Shen]

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INTRODUCTION

In 1956, Selberg expressed the trace of an invariant kernel acting on a locally symmetric space $Z = \Gamma\backslash G/K$ as a sum of certain integrals on the orbits of Γ in G, the so called "orbital integrals," and he gave a geometric expression for such orbital integrals for the heat kernel when $G = \mathrm{SL}_2(\mathbb{R})$, and the corresponding locally symmetric space is a compact Riemann surface of constant negative curvature. In this case, the orbital integrals are one to one correspondence with the closed geodesics in Z. In the general case, Harish-Chandra worked on the evaluation of orbital integrals from the 1950s until the 1970s. He could give an algorithm to reduce the computation of an orbital integral to lower dimensional Lie groups by the discrete series method. Given a reductive Lie group, in a finite number of steps, there is a formula for such orbital integrals. See Section 3.5 for a brief description of Harish-Chandra's Plancherel theory.

It is important to understand the different properties of orbital integrals even without knowing their explicit values. The orbital integrals appear naturally in Langlands program.

About 15 years ago, Bismut gave a natural construction of a Hodge theory whose corresponding Laplacian is a hypoelliptic operator acting on the total space of the cotangent bundle of a Riemannian manifold. This operator interpolates formally between the classical elliptic Laplacian on the base and the generator of the geodesic flow. We will describe recent developments in the theory of the hypoelliptic Laplacian, and we will explain two consequences of this program, the explicit formula obtained by Bismut for orbital integrals, and the recent solution by Shen of Fried's conjecture (dating back to 1986) for locally symmetric spaces. The conjecture predicts the equality of the analytic torsion and of the value at 0 of the Ruelle dynamical zeta function associated with the geodesic flow.

We will describe in more detail these two last results.

Let G be a connected reductive Lie group, let $\mathfrak{g}$ be its Lie algebra, let $\theta \in \mathrm{Aut}(G)$ be the Cartan involution of G. Let $K \subset G$ be the maximal compact subgroup of G given by the fixed-points of θ, and let $\mathfrak{k}$ be its Lie algebra. Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the corresponding Cartan decomposition of $\mathfrak{g}$.

Let B be a nondegenerate bilinear symmetric form on $\mathfrak{g}$ which is invariant under the adjoint action of G on $\mathfrak{g}$ and also under θ. We assume B is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Then $\langle \cdot, \cdot \rangle = -B(\cdot, \theta \cdot)$ is a K-invariant scalar product on $\mathfrak{g}$ that is such that the Cartan decomposition is an orthogonal splitting.

Let $C^{\mathfrak{g}} \in U(\mathfrak{g})$ be the Casimir element of G. If $\{e_i\}_{i=1}^m$ is an orthonormal basis of $\mathfrak{p}$ and $\{e_i\}_{i=m+1}^{m+n}$ is an orthonormal basis of $\mathfrak{k}$, set

(0.1)
$$B^*(\mathfrak{g}) = -\frac{1}{2} \sum_{1 \le i,j \le m} \left| [e_i, e_j] \right|^2 - \frac{1}{6} \sum_{m+1 \le i,j \le m+n} \left| [e_i, e_j] \right|^2, \quad \mathscr{L} = \frac{1}{2} C^{\mathfrak{g}} + \frac{1}{8} B^*(\mathfrak{g}).$$

Let E be a finite dimensional Hermitian vector space, let $\rho^E : K \to \mathrm{U}(E)$ be a unitary representation of K. Let $F = G \times_K E$ be the corresponding vector bundle over the symmetric space $X = G/K$. Then $\mathscr{L}$ descends to a second order differential operator $\mathscr{L}^X$ acting on $C^\infty(X, F)$. For $t > 0$, let $e^{-t\mathscr{L}^X}(x, x')$ be the smooth kernel of the heat operator $e^{-t\mathscr{L}^X}$.

Assume $\gamma \in G$ is semisimple. Then up to conjugation, there exist $a \in \mathfrak{p}, k \in K$ such that $\gamma = e^a k^{-1}$ and $\mathrm{Ad}(k)a = a$. Let $\mathrm{Tr}^{[\gamma]}\left[e^{-t\mathscr{L}^X}\right]$ denote the corresponding orbital integral of $e^{-t\mathscr{L}^X}$ (cf. (3.22), (3.46)). If $\gamma = 1$, then the orbital integral associated with $1 \in G$ is given by

$$\mathrm{Tr}^{[\gamma=1]}\left[e^{-t\mathscr{L}^X}\right] = \mathrm{Tr}^F\left[e^{-t\mathscr{L}^X}(x, x)\right] \tag{0.2}$$

which does not depend on $x \in X$.

Let $Z(\gamma) \subset G$ be the centralizer of γ, and let $\mathfrak{z}(\gamma)$ be its Lie algebra. Set $\mathfrak{p}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{p}$, $\mathfrak{k}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{k}$. Then $\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma)$.

Set $\mathfrak{z}_0 = \mathrm{Ker}(\mathrm{ad}(a)), \mathfrak{k}_0 = \mathfrak{z}_0 \cap \mathfrak{k}$. Let $\mathfrak{z}_0^\perp$ be the orthogonal space to $\mathfrak{z}_0$ in $\mathfrak{g}$. Let $\mathfrak{k}_0^\perp(\gamma)$ be the orthogonal space to $\mathfrak{k}(\gamma)$ in $\mathfrak{k}_0$, and $\mathfrak{z}_0^\perp(\gamma)$ be the orthogonal space to $\mathfrak{z}(\gamma)$ in $\mathfrak{z}_0$, so that $\mathfrak{z}_0^\perp(\gamma) = \mathfrak{p}_0^\perp(\gamma) \oplus \mathfrak{k}_0^\perp(\gamma)$. For a self-adjoint matrix Θ, set $\widehat{A}(\Theta) = \det^{1/2}\left[\frac{\Theta/2}{\sinh(\Theta/2)}\right]$. For $Y \in \mathfrak{k}(\gamma)$, set

$$(0.3)\quad J_\gamma(Y) = \left|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{-1/2} \frac{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{p}(\gamma)})}{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{k}(\gamma)})} \times \left[\frac{1}{\det(1-\mathrm{Ad}(k^{-1}))|_{\mathfrak{z}_0^\perp(\gamma)}} \frac{\det\left(1-e^{-i\,\mathrm{ad}(Y)}\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-e^{-i\,\mathrm{ad}(Y)}\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}.$$

If $\gamma = 1$, then the above equation reduces to $J_1(Y) = \frac{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{p}})}{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{k}})}$ for $Y \in \mathfrak{k} = \mathfrak{k}(1)$.

THEOREM 0.1 (Bismut's orbital integral formula [**12**], Theorem 6.1.1)
Assume $\gamma \in G$ is semisimple. Then for any $t > 0$, we have

$$(0.4)\quad \mathrm{Tr}^{[\gamma]}\left[e^{-t\mathscr{L}^X}\right] = (2\pi t)^{-\dim\mathfrak{p}(\gamma)/2} e^{-\frac{|a|^2}{2t}} \int_{\mathfrak{k}(\gamma)} J_\gamma(Y)\,\mathrm{Tr}^E\left[\rho^E(k^{-1})e^{-i\rho^E(Y)}\right] e^{-\frac{|Y|^2}{2t}} \frac{dY}{(2\pi t)^{\dim\mathfrak{k}(\gamma)/2}}.$$

There are some striking similarities of Equation (0.4) with the Atiyah-Singer index formula, where the $\widehat{A}$-genus of the tangent bundle appears. Here the $\widehat{A}$-function of both $\mathfrak{p}$ and $\mathfrak{k}$ parts (with different roles) appear naturally in the integral (0.4).

A more refined version of Theorem 0.1 for the orbital integral associated with the wave operator is given in [**12**, Theorem 6.3.2] (cf. Theorem 3.12).

Let $\Gamma \subset G$ be a discrete cocompact torsion free subgroup. The above objets constructed on X descend to the locally symmetric space $Z = \Gamma\backslash X$ and $\pi_1(Z) = \Gamma$. We denote by $\mathscr{L}^Z$ the corresponding differential operator on Z. Let $[\Gamma]$ be the set of conjugacy classes in Γ. The Selberg trace formula (cf. (3.28), (3.64)) for the heat kernel of the Casimir operator on Z says that

$$(0.5)\qquad \mathrm{Tr}[e^{-t\mathscr{L}^Z}] = \sum_{[\gamma]\in[\Gamma]} \mathrm{Vol}\left(\Gamma\cap Z(\gamma)\backslash Z(\gamma)\right)\mathrm{Tr}^{[\gamma]}[e^{-t\mathscr{L}^X}].$$

Each term $\mathrm{Tr}^{[\gamma]}[\cdot]$ in (0.5) is evaluated in (0.4).

Assume $m = \dim X$ is odd now. Let $\rho : \Gamma \to U(\mathbf{q})$ be a unitary representation. Then $F = X \times_\Gamma \mathbb{C}^{\mathbf{q}}$ is a flat Hermitian vector bundle on $Z = \Gamma\backslash X$. Let $T(F)$ be the analytic torsion associated with F on Z (cf. Definition 5.1), which is a regularized determinant of the Hodge Laplacian for the de Rham complex associated with F.

In 1986, Fried discovered a surprising relation of the analytic torsion to dynamical systems. In particular, for a compact orientable hyperbolic manifold, he identified the value at zero of the Ruelle dynamical zeta function associated with the closed geodesics in Z and with ρ, to the corresponding analytic torsion, and he conjectured that a similar result should hold for general compact locally homogenous manifolds. In 1991, Moscovici-Stanton [**54**] made an important progress in the proof of Fried's conjecture for locally symmetric spaces. The following recent result of Shen establishes Fried's conjecture for arbitrary locally symmetric spaces, and Theorem 0.1 is one important ingredient in Shen's proof.

Given $[\gamma] \in [\Gamma]\backslash\{1\}$, let $B_{[\gamma]}$ be the space of closed geodesics in Z which lie in the homotopy class $[\gamma]$, and let $l_{[\gamma]}$ be the length of the geodesic associated with γ in Z. The group $\mathbb{S}^1$ acts on $B_{[\gamma]}$ by rotations. This action is locally free. Denote by $\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]}) \in \mathbb{Q}$ the orbifold Euler characteristic number for the quotient orbifold $\mathbb{S}^1\backslash B_{[\gamma]}$. Let

$$n_{[\gamma]} = \left|\mathrm{Ker}\left(\mathbb{S}^1 \to \mathrm{Diff}(B_{[\gamma]})\right)\right| \tag{0.6}$$

be the generic multiplicity of $B_{[\gamma]}$.

THEOREM 0.2 ([**62**]). — *For any unitary representation $\rho : \Gamma \to U(\mathbf{q})$,*

$$R_\rho(\sigma) = \exp\left(\sum_{[\gamma]\in[\Gamma]\backslash\{1\}} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{n_{[\gamma]}} e^{-\sigma l_{[\gamma]}}\right) \tag{0.7}$$

is a well-defined meromorphic function on $\mathbb{C}$. If $H^\bullet(Z,F) = 0$, then $R_\rho(\sigma)$ is holomorphic at $\sigma = 0$ and

$$R_\rho(0) = T(F)^2. \tag{0.8}$$

This article is organized as follows. In Section 1, we describe Bismut's program on the geometric hypoelliptic Laplacian in de Rham theory, and we give its applications. In Section 2, we introduce the heat kernel on smooth manifolds and the basic ideas in the heat equation proof of the Lefschetz fixed-point formulas, which will serve as a model for the proof of Theorem 0.1. In Section 3, we review orbital integrals, their relation to Selberg trace formula, and we state Theorem 0.1. In Section 4, we give the basic ideas in how to adapt the construction of the hypoelliptic Laplacian of Section 1 in the context of locally symmetric spaces in order to establish Theorem 0.1. In Section 5, we concentrate on Shen's solution of Fried's conjecture.

Notation. — If A is a $\mathbb{Z}_2$-graded algebra, if $a, b \in A$, the supercommutator $[a, b]$ is given by

$$[a, b] = ab - (-1)^{\deg a \cdot \deg b} ba. \tag{0.9}$$

If B is another $\mathbb{Z}_2$-graded algebra, we denote by $A\widehat{\otimes}B$ the $\mathbb{Z}_2$-graded tensor product, such that the $\mathbb{Z}_2$-degree of $a\widehat{\otimes}b$ is given by $\deg a + \deg b$, and where the product is given by

$$(a\widehat{\otimes}b) \cdot (c\widehat{\otimes}d) = (-1)^{\deg b \cdot \deg c} ac \,\widehat{\otimes}\, bd. \tag{0.10}$$

If $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded vector space, and $\tau = \pm 1$ on $E^\pm$, for $u \in \mathrm{End}(E)$, the supertrace $\mathrm{Tr}_s[u]$ is given by

$$\mathrm{Tr}_s[u] = \mathrm{Tr}[\tau u]. \tag{0.11}$$

In what follows, we will often add a superscript to indicate where the trace or supertrace is taken.

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1. FROM HYPOELLIPTIC LAPLACIANS TO THE TRACE FORMULA

In this section, we describe some basic ideas taken from Bismut's program on the geometric hypoelliptic Laplacian and its applications to geometry and dynamical systems.

A differential operator P is hypoelliptic if for every distribution u defined on an open set U such that Pu is smooth, then u is smooth on U. Elliptic operators are hypoelliptic, but there are hypoelliptic differential operators which are not elliptic. Classical examples are Kolmogorov operator $\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$ [**44**] and Hörmander's generalization $\sum_{j=1}^k X_j^2 + X_0$ on Euclidean spaces [**42**]. Along this line, see for example Helffer-Nier's [**38**] recent book and Lebeau's work [**46**] on the hypoelliptic estimates and Fokker-Planck operators.

In 1978, Malliavin [**50**] introduced the so-called 'Malliavin calculus' to reprove Hörmander's regularity result [**42**] from a probabilistic point of view. Malliavin calculus was further developed by Bismut [**4**] and Stroock [**63**].

About 15 years ago, Bismut initiated a program whose purpose is to study the applications of hypoelliptic second order differential operators to differential geometry.

In [**6**], Bismut constructed a (geometric) hypoelliptic Laplacian on the total space of the cotangent bundle T^*M of a compact Riemannian manifold M, that depends on a parameter $b > 0$. This hypoelliptic Laplacian is a deformation of the usual Laplacian on M. More precisely, when $b \to 0$, it converges to the Laplacian on M in a suitable sense, and when $b \to +\infty$, it converges to the generator of the geodesic flow. In this way, properties of the geodesic flow on M are potentially related to the spectral properties of the Laplacian on M.

We now explain briefly Bismut's hypoelliptic Laplacian in de Rham theory. Let (M, g^{TM}) be a compact Riemannian manifold of dimension m. Let $(\Omega^\bullet(M), d)$ be the de Rham complex of M, let d^* be the formal L_2 adjoint of d, and let $\square^M = (d+d^*)^2$ be the Hodge Laplacian acting on $\Omega^\bullet(M)$.