Séminaire BOURBAKI Juin 2018
70^{e} année, 2017–2018, n^{o} 1148, p. 477 à 504
doi:10.24033/ast.1092

ON THE MONGE-AMPÈRE EQUATION

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1. INTRODUCTION

The Monge-Ampère equation is a nonlinear partial differential equation arising in several problems from analysis and geometry, such as the prescribed Gaussian curvature equation, affine geometry, optimal transportation, etc.

In its classical form, this equation is given by

$$\det D^2 u = f(x, u, \nabla u) \qquad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is some open set, $u : \Omega \to \mathbb{R}$ is a convex function, and the function $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ is given. In other words, the Monge-Ampère equation prescribes the product of the eigenvalues of the Hessian of u, in contrast with the "model" elliptic equation $\Delta u = f$ which prescribes their sum. As we shall explain later, the convexity of the solution u is a necessary condition to make the equation degenerate elliptic, and therefore to hope for regularity results.

The goal of this note is to give first a general overview of the classical theory, and then discuss some recent important developments on this beautiful topic. For our presentation of the classical theory, we follow the survey paper [**25**].

2. HISTORICAL BACKGROUND

The Monge-Ampère equation draws its name from its initial formulation in two dimensions by the French mathematicians Monge [**53**] and Ampère [**9**].

The first notable results on the existence and regularity for the Monge-Ampère equation are due to Minkowski [**51, 52**]: by approximating a general bounded convex set with convex polyhedra with given faces areas, he proved the existence of a weak solution to the "prescribed Gaussian curvature equation" (now called "Minkowski problem"). Later on, using convex polyhedra with given generalized curvatures at the vertices, Alexandrov also proved the existence of a weak solution in all dimensions, as well as the C^1 smoothness of solutions in two dimensions [**4, 5, 6**].

In higher dimension, based on his earlier works, Alexandrov [**2**] (and also Bakelman [**10**] in two dimensions) introduced a notion of generalized solution to the Monge-Ampère equation and proved the existence and uniqueness of solutions to the Dirichlet problem (see Section 3.2). The notion of weak solutions introduced by Alexandrov (now called "Alexandrov solutions") has continued to be frequently used in recent years, and a lot of attention has been drawn to prove smoothness of Alexandrov solutions under suitable assumptions on the right hand side and the boundary data.

The regularity of weak solutions in high dimensions is a very delicate problem. For $n \geq 3$, Pogorelov found a convex function in $\mathbb{R}^n$ which is not of class C^2 but satisfies the Monge-Ampère equation in a neighborhood of the origin with positive analytic right hand side (see (15) below). It became soon clear that the main issue in the lack of regularity was the presence of a line segment in the graph of u. Indeed, Calabi [**20**] and Pogorelov [**58**] were able to prove a priori interior second and third derivative estimate for strictly convex solutions, or for solutions which do not contain a line segment with both endpoints on the boundary. However, in order to perform the computations needed to deduce these a priori estimates, C^4 regularity of the solution had to be assumed. Hence, a natural way to prove existence of smooth solutions was to approximate the Dirichlet problem with nicer problems for which C^4 solutions exist, apply Pogorelov and Calabi's estimates to get C^2/C^3 a priori bounds, and then take the limit in the approximating problems. This argument was successfully implemented by Cheng and Yau [**21**] and Lions [**48**] to obtain the interior smoothness of solutions.

Concerning boundary regularity, thanks to the regularity theory developed by Ivochkina [**40**], Krylov [**45**], and Caffarelli-Nirenberg-Spruck [**19**], one may use the continuity method and Evans-Krylov's estimates [**29, 44**] to obtain globally smooth solutions to the Dirichlet problem (see Section 3.3). In particular, Alexandrov solutions are smooth up to the boundary provided all given data are smooth.

In all the situations mentioned above, one assumes that f is positive and sufficiently smooth. When f is merely bounded away from zero and infinity, Caffarelli proved the $C^{1,\alpha}$ regularity of strictly convex solutions [**14**]. Furthermore, when f is continuous (resp. $C^{0,\alpha}$), using perturbation arguments Caffarelli proved interior $W^{2,p}$ estimate for any $p > 1$ (resp. interior $C^{2,\alpha}$ estimates) [**12**].

As explained in Section 3.5, these results can be applied to obtain both the regularity in the Minkowski problem and in the optimal transportation problem. Of course, these are just some examples of possible applications of the regularity theory for Monge-Ampère. For instance, as described in the survey paper [**65**, Sections 5 and 6], Monge-Ampère equations play a crucial role in affine geometry, for example in the study of affine spheres and affine maximal surfaces.

3. CLASSICAL THEORY

In this section we give a brief overview of some relevant results on the Monge-Ampère equation. Before entering into the concept of weak solutions and their regularity, we first discuss convexity of solutions and the terminology "degenerate ellipticity" associated to this equation.

3.1. On the degenerate ellipticity of the Monge-Ampère equation

Let $u : \Omega \to \mathbb{R}$ be a smooth solution of (1) with $f = f(x) > 0$ smooth. A standard technique to prove regularity of solutions to nonlinear PDEs consists in differentiating the equation solved by u to obtain a linear second-order equation for its first derivatives. More precisely, let us fix a direction $e \in \mathbb{S}^{n-1}$ and differentiate (1) in the direction e. Then, using the formula

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \det(A+\varepsilon B) = \det(A)\,\mathrm{tr}(A^{-1}B) \qquad \forall\, A, B \in \mathbb{R}^{n\times n} \text{ with } A \text{ invertible,}$$

we obtain the equation

$$\det(D^2u)\, u^{ij}\partial_{ij}u_e = f_e \qquad \text{in } \Omega. \tag{2}$$

Here u^{ij} denotes the inverse matrix of $u_{ij} := (D^2u)_{ij}$, lower indices denotes partial derivatives (thus $u_e := \partial_e u$), and we are summing over repeated indices. Since $\det D^2u = f > 0$, the above equation can be rewritten as

$$a_{ij}\partial_{ij}u_e = \frac{f_e}{f} \qquad \text{in } \Omega, \qquad \text{where } a_{ij} := u^{ij}. \tag{3}$$

Thus, to obtain some regularity estimates on u_e, we would like the matrix a_{ij} to be positive definite in order to apply elliptic regularity theory for linear equations. But for the matrix $a_{ij} = u^{ij}$ to be positive definite we need D^2u to be positive definite, which is exactly the convexity assumption on u. [1]

We also observe that, without any a priori bound on D^2u, the matrix a_{ij} may have arbitrarily small eigenvalues and this is why one says that (1) is "degenerate elliptic".

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[1] Of course the theory would be similar if one assumes u to be concave. The real difference arises if the Hessian of u is indefinite, since (3) becomes hyperbolic (and the equation is then called "hyperbolic Monge-Ampère"). This is still a very interesting problem, but the theory for such equation is completely different from the one of the classical Monge-Ampère equation and it would go beyond the scope of this note.

Notice that if one can show that

$$c_0\mathrm{Id} \le D^2u \le C_0\mathrm{Id} \qquad \text{inside } \Omega \tag{4}$$

for some positive constants $c_0, C_0 > 0$, then $C_0^{-1}\mathrm{Id} \le (a_{ij})_{1\le i,j\le n} \le c_0^{-1}\mathrm{Id}$ and the linearized Equation (3) becomes uniformly elliptic. For this reason, proving (4) is one of the key steps for the regularity of solutions to (1).

In this regard we observe that, under the assumption $f(x) \ge \lambda > 0$, the product of the eigenvalues of D^2u (which are positive) is bounded from below. Thus, if one can prove that $|D^2u| \le C$, one easily concludes that (4) holds (see [**32**, Remark 1.1] for more details).

In conclusion, the key step towards the smoothness of solutions consists in proving that D^2u is bounded.

3.2. Alexandrov solutions

In his study of the Minkowski problem, Alexandrov introduced a notion of weak solution to the Monge-Ampère equation that allowed him to give a meaning to the Gaussian curvature of non-smooth convex sets. We now introduce this fundamental concept.

Given an open convex domain Ω, the subdifferential of a convex function $u : \Omega \to \mathbb{R}$ is given by

$$\partial u(x) := \{p \in \mathbb{R}^n \,:\, u(y) \ge u(x) + p\cdot(y-x) \quad \forall\, y \in \Omega\}.$$

One then defines the *Monge-Ampère measure* of u as follows:

$$\mu_u(E) := |\partial u(E)| \qquad \text{for every Borel set } E \subset \Omega,$$

where

$$\partial u(E) := \bigcup_{x\in E} \partial u(x)$$

and $|\cdot|$ denotes the Lebesgue measure. It is possible to show that μ_u is a Borel measure (see [**32**, Theorem 2.3]). Note that, in the case $u \in C^2(\Omega)$, the change of variable formula gives

$$|\partial u(E)| = |\nabla u(E)| = \int_E \det D^2u(x)\,dx \qquad \text{for every Borel set } E \subset \Omega,$$

therefore

$$\mu_u = \det D^2u(x)\,dx$$

(see [**32**, Example 2.2]).

This discussion motivates the following definition:

DEFINITION 3.1 (Alexandrov solutions). — *Given an open convex set* Ω *and a function* $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$, *a convex function* $u : \Omega \to \mathbb{R}$ *is called an* Alexandrov solution *to the Monge-Ampère equation*

$$\det D^2 u = f(x, u, \nabla u) \qquad \text{in } \Omega$$

if $\mu_u = f(x, u, \nabla u)\, dx$ *as Borel measures, namely*

$$\mu_u(A) = \int_A f(x, u, \nabla u)\, dx \qquad \forall\, A \subset \Omega \text{ Borel.}$$

Note that because convex functions are locally Lipschitz, they are differentiable a.e. Thus $f(x, u, \nabla u)$ is defined a.e. and the above definition makes sense.

To simplify the presentation, we shall discuss only the case $f = f(x)$, although all the arguments can be extended to the case $f = f(x, u, \nabla u)$ under the assumption that $\partial_u f \geq 0$ (this is needed to ensure that the maximum principle holds, see [**36**, Chapter 17]).

Actually, even if one is interested in solving the Monge-Ampère equation with a smooth right hand side, in order to prove existence of solutions it will be useful to consider also Borel measures as right hand sides. So, given a nonnegative Borel measure ν inside Ω, we shall say that u is an Alexandrov solution of $\det D^2 u = \nu$ if $\mu_u = \nu$.

A fundamental property of the Monge-Ampère measure is that it is stable under uniform convergence (see [**32**, Proposition 2.6]):

PROPOSITION 3.2. — *Let* $u_k : \Omega \to \mathbb{R}$ *be a sequence of convex functions converging locally uniformly to* u. *Then the associated Monge-Ampère measures* μ_{u_k} *weakly* converge to* μ_u, *i.e.,*

$$\int_\Omega \varphi\, d\mu_{u_k} \to \int_\Omega \varphi\, d\mu_u \qquad \forall\, \varphi \in C_c(\Omega).$$

Another crucial property of this definition is the validity of a comparison principle (see [**32**, Theorem 2.10]):

PROPOSITION 3.3. — *Let* $\mathcal{U} \subset \Omega$ *be an open bounded set, and let* $u, v : \Omega \to \mathbb{R}$ *be two convex functions satisfying*

$$\begin{cases} \mu_u \leq \mu_v & \text{in } \mathcal{U} \\ u \geq v & \text{on } \partial\mathcal{U}. \end{cases}$$

Then

$$u \geq v \qquad \text{in } \mathcal{U}.$$

A direct consequence of this result is the uniqueness and stability of solutions (see [**32**, Corollaries 2.11 and 2.12]):