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ESPACES ET GROUPES NON EXACTS ADMETTANT UN PLONGEMENT GROSSIER DANS UN ESPACE DE HILBERT [d'après Arzhantseva, Guentner, Osajda, Špakula]

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1. INTRODUCTION

The landscape of modern group theory has been shaped by the use of geometry as a tool for studying groups in various ways. The concept of group actions has always been at the heart of the theory, and the idea that one can link the geometry of the space on which the group acts to properties of the group, or that one can view groups as geometric objects themselves, has opened up many possibilities of interaction between algebra and geometry.

Given a finitely generated discrete group and a finite generating set of this group, one can construct an associated graph called a *Cayley graph*. This graph has the set of elements of the group as its vertex set, and two vertices are connected by an edge if one can obtain one from the other by multiplying on the right by an element of the generating set. This gives us a graph on which the group acts by isometries, viewing the graph as a metric space with the shortest path metric. While a different choice of generating set will result in a non-isomorphic graph, the two Cayley graphs will be the same up to *quasi-isometry*, a coarse notion of equivalence for metric spaces.

The study of groups from a geometric viewpoint, *geometric group theory*, often makes use of large-scale geometric information. This means that the properties of interest in this theory are often stable under small perturbations of the metric space, and it is these coarse properties that have important implications for various deep conjectures in topology and analysis. An example of this phenomenon is the celebrated work of Yu [**34**] showing that the existence of a coarse embedding (a notion of inclusion that preserves only the large-scale structure) of the Cayley graph of a finitely generated group into a Hilbert space has consequences for the coarse Baum-Connes conjecture and the strong Novikov conjecture. One way to create groups with interesting coarse geometric properties is to ensure that certain subgraphs can be found in their Cayley graphs. This can be achieved using *small cancellation theory.*

Small cancellation theory has its origins in the early twentieth century, when Dehn's work on the word problem for surface groups made small cancellation methods an important tool in algorithmic group theory. Since, small cancellation has led to the discovery of many "monster" groups, i.e., groups with pathological properties, such as the Tarski monsters of Ol'shanskii, [**23**]. Tarski monsters are infinite groups with every proper subgroup cyclic of order p for a fixed prime p. They have served as counterexamples to both the Burnside and the von Neumann-Day problems.

More recently, Gromov in [**13**] (see also [**3**]) made use of *graphical small cancellation* methods to show that there exist groups, now known as *Gromov monsters*, that are counterexamples to the Baum-Connes conjecture with coefficients [**14**]. These groups are built by encoding a sequence of finite graphs with special connectivity properties into the relations between generating elements in the group via graphical presentations—group presentations where the relators are the words that can be read along cycles in given labeled graphs. Small cancellation conditions on the labeling then ensure that the graphs are embedded in the Cayley graph of the group. Such an increasing sequence of highly-connected graphs, called an *expander*, used in Gromov's construction satisfies the somewhat contradictory properties of consisting of graphs of uniformly bounded degree and Cheeger constant bounded uniformly from below. These properties make expanders sought-after objects for applications such as cryptography or network design. Due to the presence of a weakly embedded expander in Gromov's monsters, they do not admit coarse embeddings into Hilbert spaces. Indeed, these groups were the first examples of finitely generated groups with this property.

Gromov's construction is an important example of the utility of sequences of finite graphs with exotic properties that can be used in conjunction with small cancellation machinery. The main source of such examples lies again in group theory, thanks to a way of producing graphs with desired properties as Cayley graphs of quotients of a given group. Given a residually finite group G with a fixed generating set S, we can consider a sequence of normal subgroups (N_i) of finite index with trivial intersection and study the Cayley graphs of the quotients G/N_i with respect to the generating sets induced by the images of S under the quotient maps. These graphs approximate the Cayley graph of G in a certain sense, and their coarse geometric properties can be linked to algebraic or analytic properties of the group. This allows us to use group-theoretic information to control the geometry of the resulting sequence of graphs.

An example of this is the first explicit construction of expander graphs by Margulis [**19**] using Kazhdan's property (T). Property (T) is a rigidity property of actions on Hilbert spaces—a countable group has property (T) if any affine isometric action on a Hilbert space has a fixed point. Margulis proved that a sequence of finite quotients of a group with property (T) forms an expander. Since, many such connections have been explored:

$$\begin{array}{rcl}
G \text{ is amenable} & \Longleftrightarrow & (G/N_i)_i \text{ have property A } [\mathbf{25}]; \\
G \text{ is a-(T)-menable} & \Longleftarrow & (G/N_i)_i \text{ coarsely embed into a Hilbert space } [\mathbf{25}]; \\
G \text{ has property (T)} & \Longrightarrow & (G/N_i)_i \text{ form an expander } [\mathbf{19}]; \\
G \text{ has property (T)} & \Longleftrightarrow & (G/N_i)_i \text{ have geometric property (T) } [\mathbf{31}]; \\
G \text{ has } (\tau) \text{ w.r.t. } \{N_i\} & \Longleftrightarrow & (G/N_i)_i \text{ form an expander } [\mathbf{17}].
\end{array}$$

Here, when we speak of a sequence $(G/N_i)_i$ having a certain property, we mean that the Cayley graphs of the G/N_i with respect to the images of some fixed generating set of G have this property *uniformly*. Thus, sequences of finite quotients are a rich source of examples of graphs with a variety of coarse geometric properties.

Property A, which appears above, is a non-equivariant version of amenability. For countable discrete groups, it is equivalent to exactness of the reduced C*-algebra. Such groups are referred to as *exact*. Large classes of groups, such as hyperbolic and amenable groups, have Cayley graphs that enjoy this property. Property A was first introduced by Yu in [**34**] as a way to prove coarse embeddability into a Hilbert space, in view of his above-mentioned result on the coarse Baum-Connes conjecture. Initially, it was not known whether the two properties were actually equivalent. This was first answered in the negative by Nowak [**22**], via an example that does not have bounded geometry (a metric space is said to have *bounded geometry* if for any radius, there is a uniform bound on cardinalities of balls of that radius). Nowak's example takes the form of a sequence of Cayley graphs of increasing sums of $\mathbb{Z}_2$, giving a sequence of hypercubes of increasing dimension considered with the Hamming metric.

After remaining open for some time, the question of whether there exists a bounded geometry metric space without property A that is coarsely embeddable into a Hilbert space was solved by Arzhantseva, Guentner and Špakula in [**4**], via an example of a space that distinguishes the two properties. Their construction uses finite quotients of non-amenable groups as a source of examples of spaces without property A, as described above. Arzhantseva, Guentner and Špakula's example is a carefully-chosen sequence of Cayley graphs of finite quotients of the free group on two generators F_2. The key idea in [**4**] is to use a particular sequence of nested normal subgroups which produces quotients which can be viewed as successive covering spaces with specially chosen covering groups. The covering space structure then allows them to induce *walls* on each of the quotients. The wall space structure gives rise to a metric which is coarsely equivalent to the Cayley graph metric on the quotients, and which provides a natural way to coarsely embed the graphs into a Hilbert space.

Just as amenability is an equivariant version of property A, the *Haagerup property* is the group-theoretic counterpart to coarse embeddability into a Hilbert space. A countable group is said to have the Haagerup property if it admits an affine isometric action on a Hilbert space that is metrically proper. This property is clearly incompatible with the aforementioned property (T) and is implied by amenability, and for this reason is also referred to as *a-(T)-menability*, a pun coined by Gromov.

We have the following diagram of implications between these properties, for groups.

$$\begin{array}{ccc} \text{amenability} & \Longrightarrow & \text{property A} \\ \Downarrow & & \Downarrow \\ \text{Haagerup property} & \Longrightarrow & \text{coarse embeddability into a Hilbert space} \end{array}$$

There exist examples showing that the horizontal implications are not reversible: the group $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ is non-amenable, and even has *relative property (T)* with respect to the subgroup $\mathbb{Z}^2$, but also has property A. A group G has relative property (T) with respect to a subgroup H if every affine isometric action of G on a Hilbert space has an H-fixed point. This property therefore precludes the Haagerup property if H is infinite.

The implication "amenability ⇒ Haagerup" is not reversible, since the free group F_n for $n \geq 2$ is not amenable but does admit an affine isometric action on ℓ^2. Thus, the following questions about the only remaining implications are natural: does the Haagerup property or coarse embeddability into a Hilbert space imply property A for groups? Note that the irreversibility of "property A ⇒ coarse embeddability into a Hilbert space" for metric spaces is the main result of [**4**].

As we mentioned, many classes of groups are known to have property A, and so searching for a negative answer to the above question means the rather difficult task of constructing groups without property A. For some time, the only known example had been Gromov's monster, which does not have the Haagerup property. An important step towards answering this question was taken in [**6**] by Arzhantseva and Osajda, who showed that graphical small cancellation groups on graphs with a certain walling condition have the Haagerup property. The Haagerup property had been shown for classical small cancellation groups by Wise [**32**] in the finitely presented case and by Arzhantseva and Osajda [**5**] in the infinitely presented case. Such a general result is of course not possible for graphical small cancellation groups, given that Gromov's group is in this class.

The difficulty was then to find a sequence of graphs without property A, but with an appropriate walling condition to use in the graphical presentation, while at the same time also establishing machinery that allows one to show that a small cancellation labeling exists on these graphs, in order for them to appear in the Cayley graph of the group. This was achieved in [**24**] by Osajda, using covering space methods of

[33] and **[4]** to give the required walling condition, a result of Willett **[30]** on graphs of girth (i.e., the length of the shortest cycle) tending to infinity to show that the sequence does not have property A, and the Lovász Local Lemma to prove that a small cancellation labeling exists.

Overview

In this paper, we focus on the following two results:

— There exists a bounded geometry metric space that does not have property A, but admits a coarse embedding into a Hilbert space **[4]**.

— There exists a finitely generated group that does not have property A, but admits a proper action on a CAT(0) cube complex (and has the Haagerup property) **[24]**.

In Section 2, we introduce the necessary background, including the basic ideas of geometric group theory, coarse geometry, and small cancellation theory, the relevant coarse and analytic properties, and connections of interest between group theory and geometry.

In Section 3, we give a summary of relevant results about wall spaces, embeddings, and coverings, and give a brief outline of the main result of **[4]**.

In Section 4, we summarize the construction of Osajda in **[24]**, which relies in part on previous results of Arzhantseva and Osajda **[6]**. We particularly focus on the application of the Lovász Local Lemma to create a suitable small cancellation labeling on a sequence of graphs, and methods reminiscent of those in **[4]** to induce a proper action on a CAT(0) cube complex.

2. BASIC NOTIONS

Here, we recall some basic definitions and theory necessary for the exposition of the main results.

Metric spaces from groups

The main objects of study will be finitely generated groups and their Cayley graphs. Recall that given such a group G with generating set S, the vertex set of the Cayley graph $\mathrm{Cay}(G, S)$ is the set G and the edge set is given by the pairs $\{(g, gs) : g \in G, s \in S\}$. We will refer to this Cayley graph simply as G where this does not cause confusion. The Cayley graph is a metric space with the shortest path metric, and G acts on its Cayley graph by isometries via left-multiplication.