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C*-SIMPLICITY
[after Breuillard, Haagerup, Kalantar, Kennedy and Ozawa]

by **Sven RAUM**

INTRODUCTION

Associated with a discrete group G there is the reduced group C*-algebra, defined as the closure of the complex group ring $\mathbb{C}G$ acting on the Hilbert space $\ell^2(G)$ of square summable squences on G as bounded operators by left convolution $u_g\delta_h = \delta_{gh}$. We denote the reduced group C*-algebra by $\mathrm{C}^*_{\mathrm{red}}(G) = \overline{\mathbb{C}G}^{\|\cdot\|} \subset \mathcal{B}(\ell^2 G)$. This C*-algebra naturally relates to unitary representation theory of G through the notion of weak containment of representations [**13**]. A discrete group G is called C*-simple if $\mathrm{C}^*_{\mathrm{red}}(G)$ is simple as a C*-algebra, that is every two-sided closed ideal is trivial, which translates to the property that every weakly regular unitary representation of G is automatically weakly equivalent to its regular representation. Immediately from these definitions it is clear that results on C*-simplicity can provide interesting examples of C*-algebras and they help to provide norm estimates for operators in unitary representations. Moreover, C*-simplicity can be considered a strong form of non-amenability of groups. The original motivation to study C*-simplicity was purely operator algebraic. As Pierre de la Harpe reports in [**30**, p. 13], Powers was motivated to prove simplicity of $\mathrm{C}^*_{\mathrm{red}}(\mathbb{F}_2)$ by a question due to Kaplansky: is every unital simple C*-algebra generated by its projections? Kadison's suggestion that $\mathrm{C}^*_{\mathrm{red}}(\mathbb{F}_2)$ might solve this question in the negative, led to a proof of its simplicity within two weeks, already in 1967. However, this result [**49**] was only published in 1975 and it took until 1984 when Pimser-Voiculescu could prove absence of non-trivial projections in $\mathrm{C}^*_{\mathrm{red}}(\mathbb{F}_2)$. One year earlier, Blackadar had solved Kaplansky's question in the negative by completely different methods.

After Powers published his result in 1975 until 2014, research on C*-simplicity was dominated by combinatorial methods which were formalized by Pierre de la Harpe [**29**, p. 232] in terms of the Powers property later followed by numerous weakenings and variations. Early on, it was known that normal amenable subgroups $N \trianglelefteq G$ are an obstruction to C*-simplicity, since the quotient map of groups $G \to G/N$ extends to a $*$-homomorphism $\mathrm{C}^*_{\mathrm{red}}(G) \to \mathrm{C}^*_{\mathrm{red}}(G/N)$. We refer to [**5**, Appendix G] for basic properties of amenable groups. Notably, the amenable radical of G is its maximal normal amenable subgroup. For about 30 years combinatorial methods stayed at the heart of developments in C*-simplicity, trying to address the following main problem.

Problem 1. — *Clarify the relation between the following three statements for a discrete group G.*

- *G is* C*-*simple.*
- *G has the unique trace property.*
- *The amenable radical of G is trivial.*

Here a discrete group G is said to have the unique trace property if $\mathrm{C}^*_{\mathrm{red}}(G)$ admits a unique tracial state, that is a unique linear functional $\tau : \mathrm{C}^*_{\mathrm{red}}(G) \to \mathbb{C}$ such that $\tau(x^*x) \geq 0$ and $\tau(xy) = \tau(yx)$ for all $x, y \in \mathrm{C}^*_{\mathrm{red}}(G)$. While it was clear that every C*-simple group and every group with the unique trace property must have a trivial amenable radical, it was not even known whether every C*-simple group necessarily has the unique trace property, or even any idea of a proof that C*-simplicity or the unique trace property would have concrete implications on the structure of G.

The major breakthrough in the field of C*-simplicity was obtained in the combination of work by Kalantar-Kennedy in [**37**] and by Breuillard-Kalantar-Kennedy-Ozawa in [**6**]. At the heart of this development lies the following characterisation of C*-simplicity in terms of topological dynamics.

Theorem 2 ([**37**, Theorem 1.5] and [**6**, Theorem 1.1]). — *A discrete group G is* C*-*simple if and only if its action on the Furstenberg boundary is topologically free.*

The Furstenberg boundary is a compact G-space introduced in [**19**] and featuring in a different disguise in [**26**, Remark 3]. It is introduced in Definition 20. With this characterisation at hand, Breuillard-Kalantar-Kennedy-Ozawa solved virtually all open questions on C*-simplicity in [**6**]. In particular, the unique trace property could be definitively related to the structure of the group G by the following theorem.

Theorem 3 ([**6**, Theorem 1.3]). — *A discrete group has the unique trace property if and only if its amenable radical is trivial. In particular, every* C*-*simple group has the unique trace property.*

This result, together with Le Boudec's examples of groups with trivial amenable radical that are not C*-simple [**42**], clarified all general relations in Problem 1. However, in many classes of groups, an equivalence between these three statements can be proven, and it is another major contribution of [**6**] to prove easily applicable sufficient criteria for C*-simplicity, based on the notion of normalish subgroups: a subgroup $H \leq G$ is normalish if for every $g \in G$ the set $H \cap gHg^{-1}$ is infinite.

THEOREM 4 ([**6**, Theorem 6.2]). — *A discrete group with no non-trivial finite normal subgroups and no amenable normalish subgroups is* C*-*simple.*

The present document focuses on the following tasks.

- In Section 1, we report on the notions in operator systems that led in [**37**] to the discovery of the connection between Furstenberg boundary and C*-simplicity.
- In Sections 2, 3 and 4 we report on the main achievements of [**6**] and provide a new proof for Theorem 2 which does not make any use of operator algebraic notions.
- In Section 5 we report on how these results were used by Kennedy and Haagerup in [**39, 23**] to relate back to original ideas of Powers.
- In Section 6 we summarize contemporary research related to C*-simplicity.

Given de la Harpe's exhaustive 2007 survey on C*-simplicity [**30**], we refrain from a more detailed presentation of developments in C*-simplicity before 2014, the year when the first versions of [**37, 6**] were published on arXiv. We provide a short list of references of articles on C*-simplicity between 2007 and 2014. Most of these articles addressed Problem 1 for particular classes of groups, proving equivalence between all three mentioned properties in the respective class. Linear groups were considered in [**50**], 3-manifold groups in [**31**], certain amalgamated free products in [**34**], convergence groups in [**44**] and acylindrically hyperbolic groups in [**11**]. It has to be pointed out that [**50**], following a series of revisions on arXiv, was never published, however the article's results are recovered as [**6**, Theorem 1.6] with an independent proof. A result in another direction can be found in [**46**], which provided the first examples of C*-simple groups without free subgroups, thereby showing the limits of general combinatorial ideas.

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1. BOUNDARIES

At the root of the breakthrough in C*-simplicity taking place since 2014 lies the discovery of connections between C*-algebras/representation theory on the one side and topological dynamics on the other side. While similar connections in a measurable setting, that is between von Neumann algebras and Poisson boundaries, were already successfully exploited in the past (e.g., in [**10**]), a major impact on C*-algebraic problems could only be observed after Kalantar-Kennedy [**37**] linked C*-simplicity with topological dynamics on two classical boundaries which were identified with each other: the Hamana boundary (Definition 15) and the Furstenberg boundary (Definition 20). Interestingly, this identification had already been stated by Hamana in [**26**, Remark 4], however the profound impact on our understanding of C*-simplicity had its advent in [**37**]. In this section we provide an exposition of both these boundaries, emphasizing how their major properties—injectivity and essentiality on the side of the Hamana boundary, and universality and strong proximality on the side of the Furstenberg boundary— are related to each other.

1.1. The Hamana boundary

The Hamana boundary appears as a special instance of a more general theory of injective envelopes of operator algebras developed by Hamana in the 70s in [**27, 28**], in analogy with the theory of injective hulls of modules [**14**].

Although later accounts of C*-simplicity, notably [**6**], try to avoid the notion of operator systems and we provide an operator algebra free proof of the characterisation of C*-simplicity in Section 2, the setting of operator systems was crucial for the initial discovery in [**37**] that C*-simplicity and topological dynamics are closely related. The past teaches us that operator algebras are often essential to discover new theories that later can be formulated in more elementary language. In view of this lesson and in the hope of further developments, an account on C*-simplicity would not be complete without a discussion of the ideas from operator systems underlying the discovery of [**37**].

Let us begin by recalling some basic facts on operator systems which are necessary in what follows. On a historical note, operator systems were first used in [**2**] and obtained their name in [**9**, p. 157]. Note that every unital C*-algebra (when represented on a Hilbert space) is an operator system in the sense of the following definition.

Definition 5. — *An* operator system *is a unital, self-adjoint subspace of $\mathscr{B}(H)$ for some Hilbert space H.*

Recall that an operator in $x \in \mathscr{B}(H)$ is called positive if it is self-adjoint and its spectrum lies in the positive half-line. Equivalently, $x = y^*y$ for some $y \in \mathscr{B}(H)$. An operator system $V \subset \mathscr{B}(H)$ is generated by the cone of its positive elements V^+, since $v = v^* \in V$ implies $v + \|v\|1 \in V^+$. Thus V inherits an order structure from positivity in $\mathscr{B}(H)$. The algebraic tensor products $\mathrm{M}_n(\mathbb{C}) \otimes V \subset \mathrm{M}_n(\mathbb{C}) \otimes \mathscr{B}(H)$ similarly inherit an order structure from positivity. In fact, operator systems admit an intrinsic characterisation in terms of these order structures, however here we only use it to define morphisms between operator systems.

Definition 6. — *A linear map $\varphi : V \to W$ between operator systems is called* completely positive, *if $\varphi_n : \mathrm{M}_n(\mathbb{C}) \otimes V \to \mathrm{M}_n(\mathbb{C}) \otimes W$ is positive for all $n \in \mathbb{N}_{\geq 1}$, that is the image of every positive element under φ_n is positive.*

Given a discrete group G, a G-operator system is an operator system V with an action of G by unital completely positive maps, which automatically are complete order isomorphisms. An equivariant unital completely positive map between G-operator systems is called unital completely positive G-map.

In analogy to the theory of C*-algebras, a state on an operator system V is a unital positive linear functional of norm one. The set of all states on V is denoted by $\mathscr{S}(V)$. It is a classical theorem of Stinespring that every state is a unital completely positive map [**54**]. Considering $\mathbb{C} = \mathrm{C}(\{\mathrm{pt}\})$, this generalizes to the following important correspondence between unital completely positive G-maps into commutative C*-algebras and continuous maps into the state space of an operator system. This observation provides the main connection between operator system theory and topological dynamics.

Proposition 7. — *Let V be a G-operator system and X a compact G-space. Composing a unital completely positive G-map $\Phi : V \to \mathrm{C}(X)$ with the evaluation maps $\mathrm{ev}_x : \mathrm{C}(X) \to \mathbb{C}$, $\mathrm{ev}_x(f) = f(x)$, we obtain a bijection between*

- *unital completely positive G-maps $\Phi : V \to \mathrm{C}(X)$, and*
- *G-equivariant maps $X \to \mathscr{S}(V)$.*

Let us fix the relevant notion of injectivity, which is equivalent to the categorical definition.

Definition 8. — *A G-operator system V is called* injective *if for every inclusion of G-operator systems $X \subset Y$ and any unital completely positive G-map $X \to V$ there is an extension to a unital completely positive G-map $Y \to V$.*

In view of the previous discussion on states, it is clear that the Hahn-Banach theorem implies injectivity of the operator system $\mathbb{C}$. Further, Arveson's Extension Theorem [**2**, Theorem 1.2.3] says that the operator systems $\mathscr{B}(H)$ are injective for arbitrary Hilbert spaces H.