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HOMFLY POLYNOMIALS FROM THE HILBERT SCHEMES OF A PLANAR CURVE [after D. Maulik, A. Oblomkov, V. Shende, ...]

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INTRODUCTION

Among the most interesting invariants one can associate with an oriented link $\mathscr{L} \subset S^3$ is its HOMFLY-PT polynomial $\mathbf{P}(\mathscr{L}, v, s) \in \mathbf{Z}[v^{\pm 1}, (s-s^{-1})^{\pm 1}]$ ([**12, 35**]). In 2010 A. Oblomkov and V. Shende ([**30**]) conjectured that this polynomial can be expressed in algebraic geometric terms when $\mathscr{L}$ is an *algebraic link,* that is, it is obtained as the intersection of a plane curve singularity $(C,p) \subset \mathbf{C}^2$ with a small sphere centered at p. More precisely, let $C_p^{[n]}$ be the punctual Hilbert scheme of C at p, parameterizing the length n subschemes of C supported at p. If $m : C_p^{[n]} \to \mathbf{Z}$ is the function associating with the subscheme $Z \in C_p^{[n]}$ the minimal number $m(I)$ of generators of its defining ideal I in the local ring $\mathscr{O}_{C,p}$, they conjecture that the generating function

$$Z(C,v,s) = \sum_{n \geq 0} s^{2n} \int_{C_p^{[n]}} (1-v^2)^{m(I)} d\chi_{\mathrm{top}}$$

coincides with $\mathbf{P}(\mathscr{L}, v, s)$, up to the monomial term $\left(\frac{v}{s}\right)^{\mu(f)-1}$, where $\mu(f)$ is the Milnor number. In the formula the integral is done with respect to the Euler characteristic measure $d\chi_{\mathrm{top}}$. Shortly afterwards, this surprising identity was generalized in two different directions:

1. In [**29**], Oblomkov, Rasmussen and Shende propose a "homological version": while the equality of Oblomkov and Shende is at the level of Euler characteristics, they conjecture a relation between the HOMFLY homology of Khovanov and Rozansky and the virtual Poincaré polynomial of the Hilbert schemes of (C,p). This conjecture, still open, will be shortly discussed in Section 7.
2. In [**7**], E. Diaconescu, Z. Hua and Y. Soibelman conjectured an equality in case the data of C and $\mathscr{L}$ are "colored" by choosing an array $\overrightarrow{\mu}$ of partitions, one for

every branch of the curve, or equivalently for every component of the link. This choice allows one to define two enhancements of the original objects:

- On the algebraic geometric side a thickening $C_{\overrightarrow{\mu}}$ of C, using the correspondence between partitions and monomial ideals in the plane (see Definition 5.8).
- On the link side a "satellite link" $\mathscr{L}_{\overrightarrow{\mu}} = \mathscr{L} * (Q_{\mu_1}, \dots, Q_{\mu_N})$, by associating special braids with the partitions, closing them to links Q_μ and finally wrapping them around $\mathscr{L}$ (see Eq. (61)).

In this setting, the Hilbert schemes are replaced by the moduli spaces $\mathscr{P}(Y, C, \mu, r, n)$ of stable pairs framed on $C_{\overrightarrow{\mu}}$, whereas $\mathbf{P}(\mathscr{L}, v, s)$ is replaced by the colored HOMFLY-PT polynomial $W(\mathscr{L}, \overrightarrow{\mu}; v, s)$. Diaconescu, Hua and Soibelman conjectured that a generating function arising from the topological Euler characteristic of the spaces $\mathscr{P}(Y, C, \mu, r, n)$ should coincide with $W(\mathscr{L}, {}^t\overrightarrow{\mu}; v, s)$, where ${}^t\overrightarrow{\mu}$ is the vector of transposed partitions, see Theorem 5.41 for the precise statement.

It is shown in [**29**] that choosing all partitions to be trivial yields the conjecture of Oblomkov and Shende as a special case. The conjecture of Diaconescu, Hua and Soibelman was proved by D. Maulik in 2012 in the striking paper [**23**]. The proof proceeds by showing that the two sides of the identity have the same behavior when the singular point is blown up, thus reducing to the case when the singularity is a single node, where a direct verification is possible. It is worth noticing that, even starting in the original uncolored setting of Oblomkov and Shende, the blow-up procedure leads to colored links and curves. Therefore, even though the set-up in the colored version is much more technical than the one required to explain the original conjecture, we need to discuss this level of generality, besides its intrinsic interest and beauty. As the details of the proof of Theorem 5.41 are quite involved, but well presented in the original paper [**23**], this seminar will only give a sketch of the main ideas used in the proof, and focus instead on presenting the definitions and foundations needed, along with some examples, so as to provide the necessary background for the reading of [**23**].

1. ALGEBRAIC LINKS

We summarize a few classical facts on singular points of a plane curve and their links (see [**27**] for a historical account and references to the original papers). Let (C, p) be a germ of a reduced plane curve singularity, defined as the zero set of a local equation $f = 0$, where $f \in \mathbf{C}[X, Y]$, with $f(0,0) = 0$. We denote by $\mathfrak{m}$ the maximal ideal of functions vanishing at the point $p = (0,0)$. We denote also by f and $\mathfrak{m}$ their images in $\mathbf{C}[[X,Y]]$. The point p is singular if $\partial_x f, \partial_y f \in \mathfrak{m}$. Under these hypotheses, the ideal $(\partial_x f, \partial_y f)$ is $\mathfrak{m}$-primary, and the quotient algebra $\mathbf{C}[[X,Y]]/(\partial_x f, \partial_y f)$ is a finite-dimensional vector space, whose dimension $\mu(f)$ is the *Milnor number* of the singular point. If $f = \sum_{k\in\mathbf{N}} f_k$, with f_k homogeneous of degree k, let f_d be the first nonzero homogeneous component. Then $d =: \mathrm{mult}_p(C)$ is called the *multiplicity* of C at p, and the scheme defined by $f_d(X,Y) = 0$ is the *tangent cone.* It is a union of lines, possibly with multiplicities. Let $Bl_p : \widetilde{\mathbf{A}^2(\mathbf{C})} \to \mathbf{A}^2(\mathbf{C})$ be the blow up at p. The points in the intersection of the proper transform $\widetilde{C}$ of C with the exceptional divisor correspond to the lines in the tangent cone. By the theorem on embedded resolution of singularities, [**40**, Theorem 3.4.4], there exists a sequence of blow-ups so that the (reduced) total transform of C is a normal crossing curve.

Let $f = \prod_i f_i$, with $f_i \in \mathfrak{m}$, be the factorization in irreducibles of f in $\mathbf{C}[[X,Y]]$: since C is reduced, no multiple factors appear. The curves C_i defined by the equations $f_i = 0$ are called the *branches* of the germ (C,p). Let $S^3_\epsilon \subset \mathbf{A}^2(\mathbf{C})$ be a sphere of radius ϵ centered at p. For small enough ϵ the sphere and C intersect transversally, therefore $\mathscr{L} := S^3_\epsilon \cap C$ is a nonsingular oriented one-dimensional submanifold of $S^3_\epsilon \simeq S^3$, whose isotopy class is independent of ϵ, *the link of the singularity.*

If f is irreducible in $\mathbf{C}[[X,Y]]$, then its link is connected, so actually a *knot.* More generally, the connected components of $\mathscr{L}$ correspond to the branches of (C,p).

Example 1.1. — Let $f = y^r - x^s$, with $r \leq s$. If $r < s$, the tangent cone is the line $y = 0$ with multiplicity r, while if $r = s$ it consists of the r distinct lines $y - \xi^i x = 0$, with ξ a primitive r-th root of unity. If r and s are coprime there is a unique branch, whose link is the toral (r,s) knot $\mathscr{L}_{r,s} \subset S^1 \times S^1$, parameterized by

$$x = \exp(\sqrt{-1}rt),\ y = \exp(\sqrt{-1}st) \text{ with } t \in [0, 2\pi].$$

Otherwise, let $r = da, s = db$, with a and b coprime, where d is the greatest common divisor of r and s. Letting ξ be a primitive d-th root of unity, the factorization

$$y^r - x^s = (y^a)^d - (x^b)^d = \prod_{\ell=0}^{d-1} (y^a - \xi^\ell x^b) \tag{1}$$

shows that $\mathscr{L}_{r,s}$ has d connected components, each isomorphic to the (a,b) toral knot. Notice that for $r = s = 2$ we obtain the Hopf link. As every link (Alexander's Theorem), $\mathscr{L}_{r,s}$ can be obtained as the closure of a braid: it is isomorphic to the closure of $(\beta_r)^s$, where β_r is the braid with r strands in which the first strand passes under all the other ones (if the strands are oriented from top to bottom, see Section 3.1 for the sign convention), that is, the product of the standard generators σ_i of the braid group $\mathscr{B}_r$.

Definition 1.2. — *Given two germs of curves C (resp D) through p, of equations $f=0$ (resp $g=0$), with no common factor, their* intersection number *at p is*

$$C \bullet D = \dim \mathbf{C}[[X,Y]]/(f,g). \tag{2}$$

The corresponding notion on the link side is that of linking number ([**18**, Chapter I]):

Definition 1.3. — *Given two disjoint oriented knots $K_1, K_2 \subset S^3$, let U_1 be a tubular neighborhood, homeomorphic to $S^1 \times D^2$, of K_1, disjoint from K_2. The homology group $H_1(S^3 \setminus U_1)$ is canonically isomorphic to $\mathbf{Z}$, and generated by a meridian of U_1, i.e., a circle bounding a disk in U_1 and meeting K_1 positively in only one point. Then the* linking number *of K_1 and K_2 is defined as the homology class $L(K_1,K_2) \in \mathbf{Z}$ of $K_2 \subset S^3 \setminus U_1$.*

It is easy to see that $L(K_1,K_2) = L(K_2,K_1)$ (see [**18**, Chapter I]). The relation between the two notions just defined is:

Proposition 1.4. — *If K_1 is the link of C and K_2 is the link of D, then $L(K_1,K_2) = C \bullet D$. In particular the linking numbers of components of algebraic knots are strictly positive.*

The links arising from curve singularities via this construction are called *algebraic links*, and, among their several distinctive features, probably the most important is the description of their single components as *iterated torus knots* (also called cable knots), which is the topological counterpart of the Newton–Puiseux theorem: assume that f is irreducible in $\mathbf{C}[[x,y]]$ and $f(x,y) \neq x$. Up to a change of coordinates we can assume that it is a monic polynomial in y with coefficients in $\mathbf{C}[[X]]$. Then one can "solve in y as a function of x" and the Newton–Puiseux theorem states that y can be expressed as a power series in fractional powers of x. It will be useful to write this series as

$$y(x) = x^{\frac{q_0}{p_0}}(a_0 + x^{\frac{q_1}{p_0p_1}}(a_1 + x^{\frac{q_2}{p_0p_1p_2}}(a_2 + \cdots))) \tag{3}$$

where $a_i \neq 0$, each Newton pair (p_i,q_i) consists of relatively prime positive integers, and, eventually, $p_k = 1$. This leads to an inductive description of the knot as an iterated toral knot: We consider $y(x) = x^{\frac{q_0}{p_0}}$ as the first approximation (a toral knot K_0 of type (p_0,q_0)). Then $y(x) = x^{\frac{q_0}{p_0}}(a_0 + a_1 x^{\frac{q_1}{p_0p_1}})$ gives the second approximation, describing a toral knot K_1 wrapped around K_0 and so on. In order to state this iterative description precisely, one needs at each step to have a *framing* of the knot: this notion will be discussed in a more general framework later (Section 5.2.1) and for the time being we shall limit ourselves to a "carousel" description of a specific example.

Example 1.5 ([**9**]). — Let $f(x,y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$. There is a unique branch which admits the parameterization

$$x = t^4,\ y = t^6 + t^7, \text{ or equivalently the Puiseux series } y = x^{\frac{3}{2}} + x^{\frac{7}{4}}. \tag{4}$$

Up to a rescaling, the link $\mathscr{L}$ is described by

$$x = \exp(4\sqrt{-1}t),\ \ y = \exp(6\sqrt{-1}t) + \rho \exp(7\sqrt{-1}t),$$

with $\rho \ll 1$. Since ρ is small, $\mathscr{L}$ is contained in a tubular neighborhood of the "leading knot" L of equations $x = \exp(4\sqrt{-1}t)$, $y = \exp(6\sqrt{-1}t)$ (a $(2,3)$ knot) of which $\mathscr{L}$ is a satellite: for any point of L there are two points orbiting around. In a proper parametrization of the tubular neighborhood they can be seen describing a torus knot of type $(2,13)$ (with respect to the *natural* framing, see Example 5.15).

In general the Puiseux parameterization may contain infinitely many terms, but only a finite number of them will be relevant for the topology of the knot, which will be then described as an iteration of the construction of Example 1.5, in which the types of the toric knots can be determined by the series of the Puiseux exponents [**40, 9**].

Remark 1.6. — Another important distinctive property of algebraic links is that their topology is uniquely determined by the topology of their components and their pair-wise linking numbers ([**27**, Theorem 1.1]).

2. PUNCTUAL HILBERT SCHEMES AND NESTED HILBERT SCHEMES

Given a plane curve $C \subset \mathbf{A}^2(\mathbf{C})$ and a point $p \in C$, its punctual Hilbert scheme of length n, denoted $C_p^{[n]}$, parameterizes 0-dimensional subschemes $Z \subset C$, such that $\dim \Gamma(Z, \mathcal{O}_Z) = n$ and $Z_{\rm red} = p$. Let $\mathcal{O}_{C,p}$ be the local ring of C at p and denote by $\mathfrak{m}_p$ its maximal ideal. The points $Z \in C_p^{[n]}$ will be identified with their defining ideals $I \subset \mathcal{O}_{C,p}$. The condition that $Z_{\rm red} = p$ translates into $\sqrt{I} = \mathfrak{m}_p$, and there is a natural constructible function $m : C_p^{[n]} \to \mathbf{Z}_{>0}$, defined as

$$m(I) = \text{minimal number of generators of } I = \dim_{\mathbf{C}} I/\mathfrak{m}_p I, \tag{5}$$

the last equality stemming from Nakayama's lemma.

A variant of this construction, which is relevant for our purposes, is the nested Hilbert scheme: given $\ell, n \in \mathbf{N}$,

$$C_p^{[\ell,\ell+n]} = \left\{\mathfrak{m}_p J \subset I \subset J,\ I \in C_p^{[\ell+n]}, J \in C_p^{[\ell]}\right\} \subset C_p^{[\ell+n]} \times C_p^{[\ell]}. \tag{6}$$