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A NOTE ON FROBENIUS DIVIDED MODULES IN MIXED CHARACTERISTICS

BY PIERRE BERTHELOT

ABSTRACT. — If X is a smooth scheme over a perfect field of characteristic p, and if $\mathscr{D}_X^{(\infty)}$ is the sheaf of differential operators on X [7], it is well known that giving an action of $\mathscr{D}_X^{(\infty)}$ on an $\mathscr{O}_X$-module $\mathscr{E}$ is equivalent to giving an infinite sequence of $\mathscr{O}_X$-modules descending $\mathscr{E}$ via the iterates of the Frobenius endomorphism of X [5]. We show that this result can be generalized to any infinitesimal deformation $f : X \to S$ of a smooth morphism in characteristic p, endowed with Frobenius liftings. We also show that it extends to adic formal schemes such that p belongs to an ideal of definition. In [12], dos Santos used this result to lift $\mathscr{D}_X^{(\infty)}$-modules from characteristic p to characteristic 0 with control of the differential Galois group.

RÉSUMÉ (*Une note sur les modules à descente infinie par Frobenius en caractéristique mixte*)

Si X est un schéma lisse sur un corps parfait de caractéristique p, et si $\mathscr{D}_X^{(\infty)}$ est le faisceau des opérateurs différentiels sur X [7], on sait que donner une action de $\mathscr{D}_X^{(\infty)}$ sur un $\mathscr{O}_X$-module $\mathscr{E}$ équivaut à donner une suite infinie de $\mathscr{O}_X$-modules descendant $\mathscr{E}$ par les itérés de l'endomorphisme de Frobenius de X [5]. Nous montrons que ce résultat peut être généralisé au cas d'un morphism lisse $X \to S$ qui est une déformation infinitésimale d'un morphisme de caractéristique p, munie de relèvements des morphismes de Frobenius. Nous montrons aussi qu'il s'étend aux schémas formels adiques tels que p appartienne à un idéal de définition. Ce résultat a été utilisé par dos Santos [12] pour relever les $\mathscr{D}_X^{(\infty)}$-modules de la caractéristique p à la caractéristique 0 en contrôlant le groupe de Galois différentiel du relèvement.

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Introduction

Let X_0 be a smooth scheme over a perfect field k of characteristic $p > 0$, and $\mathscr{D}^{(\infty)}_{X_0}$ the sheaf of differential operators on X_0 relative to $S_0 = \operatorname{Spec} k$ (in the sense of [7, 16.8]). A classical result of Katz [5, Th. 1.3], based on Cartier's descent [10, Th. 5.1], asserts that there is an equivalence between the category of vector bundles on X_0 endowed with a left action of $\mathscr{D}^{(\infty)}_{X_0}$, and the category of families of vector bundles $\mathscr{E}_i$ on X_0, $i \geq 0$, endowed with $\mathscr{O}_{X_0}$-linear isomorphisms $\alpha_i : F^*_{X_0}\mathscr{E}_{i+1} \xrightarrow{\sim} \mathscr{E}_i$, where F_{X_0} is the absolute Frobenius endomorphism of X_0. The purpose of this note is to explain how the theory of arithmetic $\mathscr{D}$-modules developed in [1] and [2] allows to generalize this result to infinitesimal deformations of this setup, which are not necessarily characteristic p deformations. Using limit arguments, we obtain a similar generalization for separated and complete modules over a formal scheme, including in mixed characteristics. When the base is a discrete valuation ring of mixed characteristics and $\dim(X_0) = 1$, we recover the correspondence defined earlier by Matzat [11].

We actually start with the more general situation of a smooth morphism $f_0 : X_0 \to S_0$ between characteristic p schemes. In particular, the perfection hypothesis on the basis can be removed simply by working with the relative Frobenius morphism F_{X_0/S_0} instead of the absolute Frobenius endomorphism F_{X_0} (as in [10, Th. 5.1]). We consider a nilpotent immersion $S_0 \hookrightarrow S$ and a smooth morphism $f : X \to S$ lifting f_0. We assume that an endomorphism $\sigma : S \to S$ lifting F_{S_0} and an S-morphism $F : X \to X^{(1)}$ lifting F_{X_0/S_0} are given (denoting by $X^{(i)}$ the pull-back of X by σ^i). Then our main result is Theorem 2.4, which asserts that, under these assumptions, the category of $\mathscr{D}^{(\infty)}_X$-modules is equivalent to the category of families of $\mathscr{O}_{X^{(i)}}$-modules $\mathscr{E}_i$ endowed with isomorphisms $F^*\mathscr{E}_{i+1} \xrightarrow{\sim} \mathscr{E}_i$. Note that this equivalence holds without any condition on the modules.

There are two steps in the proof. The first one is to show that, for any such family, there exists on each $\mathscr{E}_i$ a unique structure of $\mathscr{D}^{(\infty)}_{X^{(i)}}$-module such that the isomorphisms α_i are $\mathscr{D}^{(\infty)}_{X^{(i)}}$-linear (Theorem 1.2). The second one is to show that a $\mathscr{D}^{(\infty)}_X$-module can be indefinitely descended by liftings of Frobenius. While the latter is a direct consequence of the Frobenius descent theorem [2, 2.3.6], the first step is not covered by the results of [2]. It requires the whole structure provided by the infinite sequence $(\mathscr{E}_i, \alpha_i)$, but the theory of arithmetic $\mathscr{D}$-modules provides a more precise information about the differential structure obtained after a finite number of Frobenius pull-backs. Namely, the key result is the following (Proposition 1.7): if $\mathfrak{a} \subset \mathscr{O}_S$ is the ideal defining S_0, and if r is an integer such that $\mathfrak{a}^r = 0$, then, for any $m \geq 0$ and any $\mathscr{O}_{X^{(m+r)}}$-module $\mathscr{F}$, there exists on $F^{m+r\,*}\mathscr{F}$ a canonical structure of $\mathscr{D}_X^{(m)}$-module, $\mathscr{D}_X^{(m)}$ being the ring of differential operators defined in [1]. To prove the existence of this structure and its main properties, the main tool is the interpretation of $\mathscr{D}$-module structures in terms of appropriate notions of stratification, as initiated by Grothendieck [8, Appendix].

This note has been written to answer questions raised by J. P. dos Santos in his study of the variation of the differential Galois group associated to liftings of $\mathscr{D}_X^{(\infty)}$-modules from charactristic p to characteristic 0 [12]. It is a pleasure to thank him for giving me this opportunity to clarify the relations between the Frobenius descent theorem proved in [2] and the classical interpretations of $\mathscr{D}_X^{(\infty)}$-modules in terms of infinite Frobenius descent.

Conventions. — a) We denote by p a fixed prime number.

b) In this note, modules over non commutative rings will always be left modules.

1. Frobenius divided $\mathscr{D}$-modules and $\mathscr{O}$-modules

We show here that the notions of Frobenius divided $\mathscr{O}$-module and Frobenius divided $\mathscr{D}$-module coincide.

1.1. — Let S be a scheme, and $\mathfrak{a} \subset \mathscr{O}_S$ a quasi-coherent nilpotent ideal such that $p \in \mathfrak{a}$. We denote by $S_0 \subset S$ the closed subscheme defined by $\mathfrak{a}$, and we suppose given an endomorphism $\sigma : S \to S$ lifting the absolute Frobenius endomorphism of S_0. For any S-scheme X and any $i \in \mathbb{N}$, we denote by X_0 the reduction of X modulo $\mathfrak{a}$, and by $X^{(i)}$ the S-scheme deduced from X by base change by $\sigma^i : S \to S$.

We will consider S-schemes X endowed with an S-morphism $F : X \to X^{(1)}$ lifting the relative Frobenius morphism of X_0 with respect to S_0. For any $i \geq 0$, we will simply denote by $F : X^{(i)} \to X^{(i+1)}$ the morphism deduced from F by base change by σ^i. More generally, for any $i, r \geq 0$, we will denote by $F^r : X^{(i)} \to X^{(i+r)}$ the composition of the r successive morphisms $F : X^{(i+j)} \to X^{(i+j+1)}$ for $0 \leq j \leq r-1$. The morphism F is automatically finite locally free, as a consequence of the flatness criterion by fibers [6, Th. 11.3.10].

In this situation, an *F-divided $\mathscr{O}_X$-module* will be a family $(\mathscr{E}_i, \alpha_i)_{i\geq 0}$ of $\mathscr{O}_{X^{(i)}}$-modules $\mathscr{E}_i$, endowed with $\mathscr{O}_{X^{(i)}}$-linear isomorphisms $\alpha_i : F^*\mathscr{E}_{i+1} \xrightarrow{\sim} \mathscr{E}_i$. They form a category, for which the morphisms from $(\mathscr{E}_i, \alpha_i)$ to $(\mathscr{E}'_i, \alpha'_i)$ are the families of $\mathscr{O}_{X^{(i)}}$-linear homomorphisms $\mathscr{E}_i \to \mathscr{E}'_i$ which commute with the α_i's and α'_i's in the obvious sense. Note that our terminology differs a little from that of [12], where the term "F-divided" is used only when F is the actual Frobenius in characteristic p, and the term "Φ-divided" is used instead in a lifted situation of mixed characteristics. Other terminologies can be found in the literature; I hope the terminology used here will not cause any confusion.

We will assume that X is a smooth S-scheme, so that we can consider the sheaf of differential operators on X relative to S, as defined by Grothendieck in [7, 16.8]. We will denote this sheaf by $\mathscr{D}_X^{(\infty)}$, and we recall that, if $f : X \to Y$ is an S-morphism between two smooth S-schemes, the usual inverse image $f^*\mathscr{F}$ in the sense of $\mathscr{O}$-modules of a $\mathscr{D}_Y^{(\infty)}$-module $\mathscr{F}$ has a canonical structure of $\mathscr{D}_X^{(\infty)}$-module (see for example [2, 2.1.1], which is valid for $\mathscr{D}_X^{(\infty)}$-modules). Since $X^{(i)}$ is smooth over S for all i, we can introduce the notion of *F-divided $\mathscr{D}_X^{(\infty)}$-module* as being a family $(\mathscr{E}_i, \alpha_i)_{i \geq 0}$ of $\mathscr{D}_{X^{(i)}}^{(\infty)}$-modules $\mathscr{E}_i$, endowed with $\mathscr{D}_{X^{(i)}}^{(\infty)}$-linear isomorphisms $\alpha_i : F^*\mathscr{E}_{i+1} \xrightarrow{\sim} \mathscr{E}_i$.

The main result of this section is the following:

Theorem 1.2. — *Under the previous hypotheses, the obvious forgetful functor*

$$\Omega_S : \{F\text{-divided } \mathscr{D}_X^{(\infty)}\text{-modules}\} \longrightarrow \{F\text{-divided } \mathscr{O}_X\text{-modules}\} \tag{1.2.1}$$

is an equivalence of categories.

More precisely, given an F-divided $\mathscr{O}_X$-module $(\mathscr{E}_i, \alpha_i)$, there exists on each $\mathscr{E}_i$ a unique structure of $\mathscr{D}_{X^{(i)}}^{(\infty)}$-module such that the isomorphisms α_i are $\mathscr{D}_{X^{(i)}}^{(\infty)}$-linear, and each morphism of F-divided $\mathscr{O}_X$-modules is then a family of $\mathscr{D}_{X^{(i)}}^{(\infty)}$-linear maps. Endowing each $\mathscr{E}_i$ with this $\mathscr{D}_{X^{(i)}}^{(\infty)}$-module structure provides a quasi-inverse functor to Ω_S.

We will prove this statement in subsection 1.10, after a few preliminary results. We first fix the notation. If A is a commutative ring, $I \subset A$ an ideal, and $j \geq 0$ an integer, we denote by $I^{(j)} \subset A$ the ideal generated by the elements a^{p^j}, when a varies in I. Assuming that another ideal $\mathfrak{a} \subset A$ has been fixed, we define for each $i \geq 0$

$$\widetilde{I^{(i)}} := I^{(i)} + \mathfrak{a} I^{(i-1)} + \cdots + \mathfrak{a}^{i-j} I^{(j)} + \cdots + \mathfrak{a}^i I. \tag{1.2.2}$$

We will use a similar notation for sheaves of ideals.

Lemma 1.3. — *With the previous notation, assume that $p \in \mathfrak{a}$, and let $x \in \widetilde{I^{(i)}}$. Then x^p belongs to $\widetilde{I^{(i+1)}}$.*

Proof. — We can write x as $x = y_i + \cdots + y_0$, with $y_j \in \mathfrak{a}^{i-j} I^{(j)}$. Therefore,

$$x^p \in (y_i^p, \ldots, y_0^p) + p(y_i, \ldots, y_0).$$

On the one hand, we have

$$p y_j \in p\mathfrak{a}^{i-j} I^{(j)} \subset \mathfrak{a}^{i-j+1} I^{(j)} \subset \widetilde{I^{(i+1)}}.$$