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PULL-BACK OF CURRENTS BY MEROMORPHIC MAPS

BY TUYEN TRUNG TRUONG

ABSTRACT. — Let X and Y be compact Kähler manifolds, and let $f : X \to Y$ be a dominant meromorphic map. Based upon a regularization theorem of Dinh and Sibony for DSH currents, we define a pullback operator $f^\sharp$ for currents of bidegrees (p,p) of finite order on Y (and thus for *any* current, since Y is compact). This operator has good properties as may be expected.

Our definition and results are compatible to those of various previous works of Meo, Russakovskii and Shiffman, Alessandrini and Bassanelli, Dinh and Sibony, and can be readily extended to the case of meromorphic correspondences.

We give an example of a meromorphic map f and two nonzero positive closed currents T_1, T_2 for which $f^\sharp(T_1) = -T_2$. We use Siu's decomposition to help further study on pulling back positive closed currents. Many applications on finding invariant currents are given.

RÉSUMÉ (*Pull-back de courants par des applications méromorphes*)

Soient X et Y des variétés kählériennes compactes, et $f\colon X \to Y$ une application méromorphe dominante. En nous basant sur un théorème de régularisation de Dinh et Sibony pour des courants DSH, nous définissons un opérateur pullback $f^\sharp$ pour les courants de bidegré (p,p) d'ordre fini sur Y (et donc pour *tout* courant, puisque Y est compact. Cet opérateur a des bonnes propriétés, comme attendu.

Notre définition et nos résultats sont compatibles avec ceux des travaux précédents de Meo, Russakovskii et Shiffman, Alessandrini et Bassanelli, Dinh et Sibony, et peut être facilement étendu au cas des correspondances méromorphes.

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Nous donnons un exemple d'application méromorphe f et deux courants fermés positifs non-nuls T_1, T_2 pour lesquels $f^\sharp(T_1) = -T_2$. Nous utilisons la décomposition de Siu pour faciliter l'étude des courants fermés positifs pullback. Nous donnons une multitude d'applications autour de la recherche de courants invariants.

1. Introduction

Let X and Y be two compact Kähler manifolds, and let $f : X \rightarrow Y$ be a dominant meromorphic map. For a (p,p)-current T on Y, we seek to define a pullback $f^\sharp(T)$ which has good properties. Such a pullback operator will be helpful in complex dynamics, in particular in the problem of finding invariant closed currents for a selfmap.

We let $\pi_X, \pi_Y : X \times Y \rightarrow X, Y$ be the two projections (When $X = Y$ we denote these maps by π_1 and π_2). Let $\Gamma_f \subset X \times Y$ be the graph of f, and let $\mathscr{C}_f \subset \Gamma_f$ be the critical set of π_Y, i.e., the smallest analytic subvariety of Γ_f so that the restriction of π_Y to $\Gamma_f - \mathscr{C}_f$ has fibers of dimension $dim(X) - dim(Y)$. For a set $B \subset Y$, we define $f^{-1}(B) = \pi_X(\pi_Y^{-1}(B) \cap \Gamma_f)$, and for a set $A \subset X$ we define $f(A) = \pi_Y(\pi_X^{-1}(A) \cap \Gamma_f)$.

If T is a smooth form on Y, then it is standard to define $f^*(T)$ as a current on X by the formula $f^*(T) = (\pi_X)_*(\pi_Y^*(T) \wedge [\Gamma_f])$. This definition descends to cohomology classes: If T_1 and T_2 are two closed smooth forms on Y having the same cohomology classes, then $f^*(T_1)$ and $f^*(T_2)$ have the same cohomology class in X. This allows us to define a pullback operator on cohomology classes. These considerations apply equally to continuous forms. When T is an arbitrary current on Y, we can still define $\pi_Y^*(T)$ as a current on $X \times Y$. However, in general it is not known how to define the wedge product of the two currents $\pi_Y^*(T)$ and $[\Gamma_f]$. This is the source of difficulty for defining pullback for a general current.

For some important classes of currents (positive closed and positive dd^c-closed currents, DSH currents, for definitions see the next subsection), there have been works on this topic by Meo [15], Russakovskii and Shiffman [16], Alessandrini and Bassanelli [1], Dinh and Sibony [10],[11]. We will give more details on these works later, but here will discuss only some general ideas used in these papers. Roughly speaking, in the works cited above, to define pullback of a (p,p) current T, the authors use approximations of T by sequences of smooth (p,p) forms T_n satisfying certain properties, and then define $f^\sharp(T) = \lim_{n\rightarrow\infty} f^*(T_n)$ if the limit exists and is the same for all such sequences. In order to have such approximations then T must have some positive property. In these definitions, the resulting pullback of a positive current is again positive.

Our idea for pulling back a general (p,p) current T is as follows. Assume that we have a well-define pullback $f^\sharp(T)$. Then for any smooth form of complement bidegree α we should have

$$\int_X f^\sharp(T)\wedge\alpha = \int_Y T\wedge f_*(\alpha).$$

The wedge product in the integral of the RHS is not well-defined in general. To define it we adapt the above idea, that is to use smooth approximations of either T or $f_*(\alpha)$. Fortunately, since Y is compact, any current T is of a finite order s. Moreover since $f_*(\alpha)$ is a DSH current, we can use the regularization theorem in [11] to produce approximation by C^s forms $K_n(f_*(\alpha))$ with desired properties. Then we define

$$\int_X f^\sharp(T)\wedge\alpha = \lim_{n\to\infty}\int_Y T\wedge K_n(f_*(\alpha)),$$

if the limit exists and is the same for such good approximations. The details of this definition will be given in the next subsection. We conclude this subsection commenting on the main results of this paper:

-Our pullback operator is compatible with the standard definition for continuous form and with the definitions in the works cited above.

-There are examples of losing positivity for currents of higher bidegrees when pulled back by meromorphic maps.

-We obtain a natural criterion on pulling back analytic varieties which, combined with Siu's decomposition, can be used to help further study pullback of general positive closed currents.

-We can apply the definition to examples having invariant positive closed currents of higher bidegrees whose supports are contained in pluripolar sets.

1.1. Definitions. — For convenience, let us first recall some facts about currents. The notations of positive and strongly positive currents in this paper follow the book [6]. For a current T on Y, let $supp(T)$ denote the support of T. Given $s\geq 0$, a current T is of order s if it acts continuously on the space of C^s forms on Y equipped with the usual C^s norm. A positive (p,p) current T is of order 0. If T is a positive (p,p) current then its mass is defined as $||T|| =< T,\omega_Y^{dim(Y)-p} >$, where ω_Y is a given Kähler $(1,1)$ form of Y. If T is a closed current on Y, we denote by $\{T\}$ its cohomology class. If V is a subvariety in Y, we denote by $[V]$ the current of integration on V, which is a strongly positive closed current. We use $\rightharpoonup$ for weak convergence of currents.

For any p, we define $DSH^p(Y)$ (see Dinh and Sibony [8]) to be the space of (p,p) currents $T=T_1-T_2$, where T_i are positive currents, such that $dd^cT_i = \Omega_i^+ - \Omega_i^-$ with $\Omega_i^\pm$ positive closed. Observe that $||\Omega_i^+|| = ||\Omega_i^-||$ since they are cohomologous to each other because $dd^c(T_i)$ is an exact current. Define the DSH-norm of T as

$$||T||_{DSH} := \min\{||T_1|| + ||T_2|| + ||\Omega_1^+|| + ||\Omega_2^+||,\ T_i,\ \Omega_i,\ \text{as above}\}.$$

Using compactness of positive currents, it can be seen that we can find T_i, $\Omega_i^{\pm}$ which realize $||T||_{DSH}$, hence the minimum on the RHS of the definition of DSH norm. We say that $T_n \rightharpoonup T$ in $DSH^p(Z)$ if T_n weakly converges to T and $||T_n||_{DSH}$ is bounded.

Our definition is modelled on the smooth approximations given by Dinh and Sibony [8]. However, some restrictions should be imposed on the approximations when we deal with the case of general maps:

1) Since any definition using local approximations will give a positive current as the resulting pullback of positive currents, in general we need to use global approximations in order to deal with the cases like the map J_X in Section 4.

2) For a general compact Kähler manifold, it is not always possible to approximate a positive closed current by positive closed smooth forms (see Proposition 2 for an example where even the negative parts of the approximation are not bounded).

3) The more flexible we allow in approximating currents, the more restrictive the maps and currents we can define pullback. For example, we have the following observation

Lemma 1. — *Assume that for any positive closed smooth (p,p) form T and for every sequence of positive closed smooth forms $T_n^{\pm}$ whose masses $||T_n^{\pm}||$ are uniformly bounded and $T_n^+ - T_n^- \rightharpoonup T$, then $f^*(T_n^+ - T_n^-) \rightharpoonup f^*(T)$. Then the same property holds for any positive closed (p,p) current T.*

Proof. — In fact, let $T_n^+ - T_n^-$ and $S_n^+ - S_n^-$ be two sequences weakly converging to a positive closed (p,p) current T, where $T_n^{\pm}$ and $S_n^{\pm}$ are positive closed smooth (p,p) forms having uniformly bounded masses. Then $(T_n^+ + S_n^-) - (T_n^- + S_n^+)$ is a sequence weakly converges to 0 with the same property, and because 0 is a smooth form, we must have $f^*(T_n^+ + S_n^-) - f^*(T_n^- + S_n^+)$ weakly converges to 0 by assumption. Hence $f^*(T_n^+ - T_n^-)$ and $f^*(S_n^+ - S_n^-)$ converges to the same limit. □

Roughly speaking, under the conditions of Lemma 1 then all positive closed currents can be pulled back. However, this is not true in general (see Example 2). We will restrict to use only good approximation schemes, defined as follows

Definition 1. — *Let Y be a compact Kähler manifold. Let $s \geq 0$ be an integer. We define a good approximation scheme by C^s forms for DSH currents on Y to be an assignment that for a DSH current T gives two sequences $K_n^{\pm}(T)$ (here $n = 1, 2, \dots$) where $K_n^{\pm}(T)$ are C^s forms of the same bidegrees as T, so*