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PERTURBATIONS OF FLEXIBLE LATTÈS MAPS

BY XAVIER BUFF & THOMAS GAUTHIER

ABSTRACT. — We prove that any Lattès map can be approximated by strictly postcritically finite rational maps which are not Lattès maps.

RÉSUMÉ (*Perturbations des exemples de Lattès flexibles*). — Nous montrons que tout exemple de Lattès peut être approché par des fractions rationnelles strictement postcritiquement finies qui ne sont pas des exemples de Lattès.

Introduction

A rational map of degree $D \geq 2$ is *strictly postcritically finite* if the orbit of each critical point intersects a repelling cycle. Among those, flexible Lattès maps (the definition is given below) play a special role. The following result answers a question raised in [4].

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Theorem. — *Every flexible Lattès map can be approximated by strictly postcritically finite rational maps which are not Lattès maps.*

Given $D \geq 2$, denote by Rat_D the space of rational maps of degree D. A rational map $f \in \mathrm{Rat}_D$ has a *Julia set* J_f which may be defined as the closure of the set of repelling cycles of f. The Julia set is the support of a measure μ_f which may be defined as the unique invariant measure of (maximal) entropy $\log D$.

The *bifurcation locus* in Rat_D is the closure of the set of discontinuity of the map $f \mapsto J_f$. Laura DeMarco [5] proved that the bifurcation locus is the support of a positive closed $(1,1)$-current $T_{\mathrm{bif}}{:=}dd^c\mathfrak{L}$, where $\mathfrak{L}$ is the plurisubharmonic function which sends a rational map f to its Lyapunov exponent with respect to μ_f.

Möbius transformations act by conjugacy on Rat_D and the quotient space is an orbifold known as the *moduli space* $\mathcal{M}_D$ of rational maps of degree D. Giovanni Bassanelli and François Berteloot [1] introduced a measure μ_{bif} on this moduli space, which may be obtained by pushing forward $T_{\mathrm{bif}}^{\wedge(2D-2)}$.

In [4], the first author and Adam Epstein, using a transversality result in Rat_D, proved that the conjugacy class of a strictly postcritically finite map which is not a flexible Lattès map is in the support of μ_{bif}. Since the support of μ_{bif} is closed and since the conjugacy class of a strictly postcritically finite rational maps which is not a Lattès maps is in this support, our result has the following consequence.

Corollary 1. — *The classes of flexible Lattès maps in $\mathcal{M}_D$ lie in the support of the bifurcation measure μ_{bif}.*

In [7], the second author proved that the support of the bifurcation measure has maximal Hausdorff dimension, i.e. has dimension $2(2D-2)$, and that it is homogeneous (the support of μ_{bif} has maximal dimension in any neighborhood of its points). Corollary 1 thus yields the following result.

Corollary 2. — *Let $f \in \mathrm{Rat}_D$ be a flexible Lattès map and let $V \subset \mathcal{M}_D$ be an open neighborhood of the conjugacy class of f. Then,* $\dim_H(\mathrm{supp}(\mu_{\mathrm{bif}}) \cap V) = 2(2D-2)$.

Bassanelli and Berteloot [2] proved that every point in the support of μ_{bif} can be approximated by rational maps having $2D-2$ distinct neutral cycles. Their argument can be adapted to prove that the support of μ_{bif} can be approximated by hyperbolic maps having $2D-2$ distinct attracting cycles (see [3] Section 6.2). By Corollary 1, we have the following result.

Corollary 3. — *Any flexible Lattès map $f \in \mathrm{Rat}_D$ can be approximated by hyperbolic rational maps having $2D-2$ distinct attracting cycles.*

The approach for solving this problem was suggested by John Milnor. We wish to express our gratitude. We also wish to thank the Banff International Research Station for hosting the workshop "Frontiers in Complex Dynamics (Celebrating John Milnor's 80th birthday)" during which we developed our proof.

1. Flexible Lattès maps

Following Milnor [8], we define a *flexible Lattès map* of degree $D \geq 2$ to be a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for which there is a commutative diagram :

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{L} & \mathbb{C}/\Lambda \\ \Theta \downarrow & & \downarrow \Theta \\ \widehat{\mathbb{C}} & \xrightarrow[f]{} & \widehat{\mathbb{C}} \end{array}$$

where

- $\Lambda \subset \mathbb{C}$ is a lattice of rank 2;
- $\mathcal{T}:=\mathbb{C}/\Lambda$ is the quotient torus;
- $L : \mathcal{T} \ni \tau \mapsto a\tau + b \in \mathcal{T}$ with $a \in \mathbb{Z}$, $a^2 = D$, and $2b \in \Lambda/\big(2\Lambda + (a-1)\Lambda\big)$;
- $\Theta : \mathcal{T} \to \widehat{\mathbb{C}}$ is a 2-to-1 holomorphic map ramifying at points in $\Lambda/2$.

Conjugating L with an affine map if necessary, we may assume $\Lambda = \mathbb{Z} \oplus \gamma\mathbb{Z}$ where γ is a complex number in the upper half-plane $\mathbb{H}$ and that we are in one of the following three cases.

- Case 1 : a is even. In that case $L(\tau) = a\tau$.
- Case 2 : a is odd and $2b = 0 \in \Lambda/(2\Lambda)$. In that case $L(\tau) = a\tau$.
- Case 3 : a is odd and $2b \neq 0 \in \Lambda/(2\Lambda)$. In that case we may choose γ so that $L(\tau) = a\tau + \frac{\gamma+1}{2}$.

In addition, conjugating f with a Möbius transformation, we may assume that $\Theta(0) = 0$, $\Theta\left(\frac{\gamma+1}{2}\right) = \infty$ and $\Theta\left(\frac{1}{2}\right) = 1$.

In the rest of the article, the lattice Λ will be of the form $\Lambda_\gamma := \mathbb{Z} \oplus \gamma\mathbb{Z}$, where γ is a complex number which is allowed to vary in the upper half-plane $\mathbb{H}$ of complex numbers with positive imaginary part. We shall denote by $\Theta_\gamma : \mathcal{T}_\gamma \to \widehat{\mathbb{C}}$ the degree 2 covering map which ramifies at the points in $\Lambda/2$, normalized by the conditions:

$$\Theta_\gamma(0) = 0, \quad \Theta_\gamma\left(\frac{\gamma+1}{2}\right) = \infty, \quad \Theta_\gamma\left(\frac{1}{2}\right) = 1 \quad \text{and} \quad \Theta_\gamma\left(\frac{\gamma}{2}\right) = w(\gamma).$$

The function $w : \mathbb{H} \to \mathbb{C} - \{0\}$ is holomorphic. In order to have a more symmetric presentation, we let $v : \mathbb{H} \to \mathbb{C} - \{0\}$ be the constant function equal to 1 and note that for all $\gamma \in \mathbb{H}$, $v(\gamma) \neq w(\gamma)$.

The derivative of the torus endomorphism $L_\gamma : \mathcal{T}_\gamma \to \mathcal{T}_\gamma$ will be a fixed integer a which does not depend on γ. When $a = \pm 2$, the critical value set of the Lattès map $f_\gamma : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is $\{\infty, v(\gamma), w(\gamma)\}$ and if $|a| \geq 3$, the critical value set is $\{0, \infty, v(\gamma), w(\gamma)\}$. In all cases, the postcritical set of f_γ is $\{0, \infty, v(\gamma), w(\gamma)\}$.

More precisely, we will have the following dynamics on the postcritical set :

- Case 1 : all the critical values are mapped to 0 which is a fixed point of f.
- Case 2 : all the critical values are fixed with multiplier a^2.
- Case 3 : the Lattès map permutes the critical value at 0 with that at infinity. It also permutes the critical value at $v(\gamma)$ with the critical value at $w(\gamma)$. The multiplier of each cycle is a^4.

From now on, we assume that we are in one of those three cases, and we consider the analytic family of Lattès maps

$$\mathbb{H} \ni \gamma \mapsto f_\gamma \in \mathrm{Rat}_D,$$

where Rat_D is the space of rational maps of degree D. We shall use the notation f, v, w, ... in place of f_γ, $v(\gamma)$, $w(\gamma)$, ... when γ is assumed to be fixed and there is no confusion.

2. Estimates for Θ

Lemma 1. — *As $\tau \to 0$, we have the following expansion*

$$\Theta(1/2 + \tau) = v + \lambda\tau^2 + o(\tau^2) \quad \textit{and} \quad \Theta(\gamma/2 + \tau) = w + \mu\tau^2 + o(\tau^2)$$

with

$$\frac{\lambda}{v} = -\frac{\mu}{w} \neq 0.$$

Proof. — Since Θ has simple critical points at $1/2$ and $\gamma/2$, we have an expansion as in the statement with $\lambda \neq 0$ and $\mu \neq 0$. Our work consists in proving the relation between λ/v and μ/w. Let q be the meromorphic quadratic differential on $\widehat{\mathbb{C}}$ defined by :

$$q := \frac{dz^2}{z(z-v)(z-w)}.$$

Since q has simple poles and since Θ is totally ramified above the polar set of q, the quadratic differential $\Theta^* q$ is holomorphic on $\mathcal{T}$, whence

$$\Theta^* q = \kappa \cdot d\tau^2 \quad \text{with} \quad \kappa \in \mathbb{C} - \{0\}.$$