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L^p ESTIMATES FOR MULTI-LINEAR AND MULTI-PARAMETER PSEUDO-DIFFERENTIAL OPERATORS

by Wei Dai & Guozhen Lu

Abstract. — We establish the pseudo-differential variant of the L^p estimates for multi-linear and multi-parameter Coifman-Meyer multiplier operators proved by C. Muscalu, J. Pipher, T. Tao and C. Thiele in [21, 22]. This gives an affirmative answer to the question, raised in the book of C. Muscalu and W. Schlag [23], on whether the L^p estimates for multi-linear and multi-parameter pseudo-differential operators hold.

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1. Introduction

1.1. Background. — For $n \geq 1$ and $d \geq 1$, let m be a bounded function in $\mathbb{R}^{nd}$, smooth away from the origin and satisfying Hörmander-Mikhlin conditions [(1)]

$$|\partial^{\alpha} m(\xi)| \lesssim \frac{1}{|\xi|^{|\alpha|}} \tag{1.1}$$

for sufficiently many multi-indices α. Denote by T_m the n-linear operator defined by

$$T_m(f_1, \dots, f_n)(x) := \int_{\mathbb{R}^{nd}} m(\xi) \hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) e^{2\pi i x\cdot(\xi_1+\cdots+\xi_n)} d\xi, \tag{1.2}$$

where $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^{nd}$ and $f_1, \dots, f_n$ are Schwartz functions on $\mathbb{R}^d$. From the classical Coifman-Meyer theorem (see [6, 7, 19, 11, 15]), we know that the operator T_m extends to a bounded n-linear operator from $L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_n}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$, provided that $1 < p_1, \dots, p_n \leq \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_n} > 0$. When $n = 2$, as a consequence of bilinear $T1$ theorem (see [6, 11]), there is also a pseudo-differential variant of the classical Coifman-Meyer theorem for symbol $a \in BS^0_{1,0}(\mathbb{R}^{3d})$, that is, a satisfies the differential inequalities

$$|\partial_x^{\gamma} \partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} a(x, \xi, \eta)| \lesssim_{d,\alpha,\beta,\gamma} (1 + |\xi| + |\eta|)^{-|\alpha|-|\beta|} \tag{1.3}$$

for sufficiently many multi-indices α, β, γ. Namely, let T_a be the corresponding bilinear pseudo-differential operators defined by replacing m with a in (1.2), then T_a is bounded from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ into $L^r(\mathbb{R}^d)$, provided that $1 < p, q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} > 0$ (see [2], and see [3, 26, 23] for $d = 1$ case). For large amounts of literature involving estimates for multi-linear Calderón-Zygmund operators and multi-linear pseudo-differential operators, refer to e.g., [1, 6, 19, 9, 11, 12, 15, 23, 24].

However, when we come into the situation that a differential operator (with different behaviors on different spatial variables x_i, $i = 1, \dots, d$) acts on a product of several functions (for instance, the bilinear form $\mathscr{D}_1^{\alpha} \mathscr{D}_2^{\beta}(fg)$, where $\widehat{\mathscr{D}_1^{\alpha} f}(\xi_1, \xi_2) := |\xi_1|^{\alpha} \hat{f}(\xi_1, \xi_2)$ and $\widehat{\mathscr{D}_2^{\beta} f}(\xi_1, \xi_2) := |\xi_2|^{\beta} \hat{f}(\xi_1, \xi_2)$ for α, $\beta > 0$), we realize that the necessity to investigate bilinear and bi-parameter operators

(1) Throughout this paper, $A \lesssim B$ means that there exists a universal constant $C > 0$ such that $A \leq CB$. If necessary, we use explicitly $A \lesssim_{\star,\dots,\star} B$ to indicate that there exists a positive constant $C_{\star,\dots,\star}$ depending only on the quantities appearing in the subscript continuously such that $A \leq C_{\star,\dots,\star} B$.

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$T_m^{(2)}$ defined by

$$T_m^{(2)}(f,g)(x) := \int_{\mathbb{R}^4} m(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta, \tag{1.4}$$

where the symbol m is smooth away from the planes $(\xi_1,\eta_1) = (0,0)$ and $(\xi_2,\eta_2) = (0,0)$ in $\mathbb{R}^2 \times \mathbb{R}^2$ and satisfying the less restrictive Marcinkiewicz conditions

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\beta_1}\partial_{\eta_1}^{\alpha_2}\partial_{\eta_2}^{\beta_2} m(\xi,\eta)| \lesssim \frac{1}{|(\xi_1,\eta_1)|^{\alpha_1+\alpha_2}} \cdot \frac{1}{|(\xi_2,\eta_2)|^{\beta_1+\beta_2}} \tag{1.5}$$

for sufficiently many multi-indices $\alpha = (\alpha_1,\alpha_2)$, $\beta = (\beta_1,\beta_2)$. It becomes more complicated and difficult to establish the L^p estimates for $T_m^{(2)}$ than in the one-parameter multilinear situations or L^p estimates for linear multi-parameter singular integrals (see e.g., [8] and [14]). In [21], by using the duality lemma of $L^{p,\infty}$ presented in [24], the $L^{1,\infty}$ sizes and energies technique developed in [25] and multi-linear interpolation (see e.g., [10, 25]), Muscalu, Pipher, Tao and Thiele proved the following L^p estimates for $T_m^{(2)}$ (see also [23], and for subsequent endpoint estimates see [16]).

Theorem 1.1 ([21]). — *The bilinear operator $T_m^{(2)}$ defined by* (1.4) *maps $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \to L^r(\mathbb{R}^2)$ boundedly, as long as $1 < p,q \le \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} > 0$.*

In general, any collection of n generic vectors $\xi_1 = (\xi_1^i)_{i=1}^d, \ldots, \xi_n = (\xi_n^i)_{i=1}^d$ in $\mathbb{R}^d$ generates naturally the following collection of d vectors in $\mathbb{R}^n$:

$$\bar{\xi}_1 = (\xi_j^1)_{j=1}^n, \quad \bar{\xi}_2 = (\xi_j^2)_{j=1}^n, \quad \ldots, \quad \bar{\xi}_d = (\xi_j^d)_{j=1}^n. \tag{1.6}$$

Let $m = m(\xi) = m(\bar{\xi})$ be a bounded symbol in $L^\infty(\mathbb{R}^{dn})$ that is smooth away from the subspaces $\{\bar{\xi}_1 = 0\} \cup \cdots \cup \{\bar{\xi}_d = 0\}$ and satisfying

$$|\partial_{\bar{\xi}_1}^{\alpha_1} \cdots \partial_{\bar{\xi}_d}^{\alpha_d} m(\bar{\xi})| \lesssim \prod_{i=1}^d |\bar{\xi}_i|^{-|\alpha_i|} \tag{1.7}$$

for sufficiently many multi-indices $\alpha_1,\ldots,\alpha_d$. Denote by $T_m^{(d)}$ the n-linear multiplier operator defined by

$$T_m^{(d)}(f_1,\ldots,f_n)(x) := \int_{\mathbb{R}^{dn}} m(\xi)\hat{f}_1(\xi_1)\cdots\hat{f}_n(\xi_n)e^{2\pi i x\cdot(\xi_1+\cdots+\xi_n)}d\xi. \tag{1.8}$$

In [22], Muscalu, Pipher, Tao and Thiele generalized Theorem 1.1 to the n-linear and d-parameter setting for any $n \ge 1$, $d \ge 2$, their result is stated in the following theorem (see also [23]).