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SIGNATURE SPECTRUM OF POSITIVE BRAIDS

by Sebastian Baader

Abstract. — We derive a lower bound for all Levine–Tristram signatures of positive braid links, linear in terms of the first Betti number. As a consequence, we obtain upper and lower bounds on the ratio of fixed pairs of Levine–Tristram signature invariants, valid uniformly on all positive braid monoids.

Résumé (*Signatures de Levine-Tristram associées aux tresses positives*). — Nous dérivons une borne inférieure pour les signatures de Levine-Tristram associées aux tresses positives. Cette borne étant proportionnelle au nombre de Betti de la surface de Seifert canonique, nous obtenons également une borne pour le rapport des invariants de Levine-Tristram des tresses positives, pour tous pairs de nombres du cercle unité.

1. Introduction

Around 40 years ago, Rudolph showed that knotted positive braid links have strictly positive signature invariants and concluded that these links are not slices [12]. The closure $\hat{\beta} \subset S^3$ of a non-split positive braid $\beta \in B_n$ with n strings and k crossings bounds a canonical genus-minimising Seifert surface $\Sigma(\beta) \subset S^3$, whose first Betti number is $b_1(\Sigma(\beta)) = k - n$ [3]. Feller improved

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Rudolph's positivity result to the following linear bound for the signature invariant:

$$\sigma(\beta) \geq \frac{1}{8} b_1(\beta)$$

(see [7] for this bound and [6] for the first linear bound of this type, both by Feller). In this note, we derive a linear lower bound for the family of Levine–Tristram signature invariants $\sigma_\omega(L)$, parametrised by $\omega \in S^1$, defined from any Seifert matrix V of a link L as the signature of the Hermitian form

$$M_\omega = (1-\omega)V + (1-\bar{\omega})V^T.$$

For the class of links considered here – closures of non-split positive braids – this form is non-degenerate, provided that ω is not an algebraic number, i.e. $\omega \in S^1 \setminus \bar{\mathbb{Q}}$. Indeed, the Alexander polynomial of non-split positive braid links is non-zero [4]; this implies $\det(M_\omega) \neq 0$, for all $\omega \in S^1 \setminus \bar{\mathbb{Q}}$. In particular, the signature is just the difference of the numbers of positive and negative eigenvalues of the matrix M_ω (see [11, 15] for the original definition and [5] for an excellent survey on the Levine–Tristram signature invariants). We use the notation B_n^+ for the monoid of non-split positive braids on n strands and $\sigma_\omega(\beta)$ for the Levine–Tristram signature invariant of the closure of a braid $\beta \in B_n^+$.

Theorem 1.1. — *For all $\omega \in S^1 \setminus \bar{\mathbb{Q}}$, there exists a constant $c > 0$, such that for all $n \in \mathbb{N}$:*

$$\liminf \left\{ \frac{\sigma_\omega(\beta)}{b_1(\beta)} \middle| \beta \in B_n^+ \right\} \geq c.$$

Here, the limit inferior stands for the smallest accumulation point of the set of ratios $\frac{\sigma_\omega(\beta)}{b_1(\beta)}$. We cannot replace this by the mere infimum, since the piecewise constant signature function of a knot always starts off with a zero segment. While each individual of these bounds may not appear that impressive, we would like to record the following consequence of Theorem 1.1 and the inequalities $\sigma_\omega(\beta) \leq 2b_1(\beta)$, valid for all $\omega \in S^1 \setminus \bar{\mathbb{Q}}$ and for all positive braids β.

Corollary 1.2. — *Let $\omega_1, \omega_2 \in S^1 \setminus \bar{\mathbb{Q}}$. There exist constants $0 < a < b$, so that the following inequalities hold for all $n \geq 2$:*

$$a \leq \liminf \left\{ \frac{\sigma_{\omega_1}(\beta)}{\sigma_{\omega_2}(\beta)} \middle| \beta \in B_n^+ \right\} \leq \limsup \left\{ \frac{\sigma_{\omega_1}(\beta)}{\sigma_{\omega_2}(\beta)} \middle| \beta \in B_n^+ \right\} \leq b.$$

Here again, we only consider the signature invariant for non-algebraic ω, since we do not want to deal with complications caused by nullity, which occurs if $\det(M_\omega) = 0$. The statement of the corollary can be seen as a coarse version of the fact that all homomorphisms from the braid group B_n to $\mathbb{R}$ are proportional. It also complements the existence of the asymptotic ratio $\frac{\sigma_{\omega_1}}{\sigma_{\omega_2}}$ for limits of orbits of ergodic vector fields on the 3-sphere [1]. The proof of Theorem 1.1

has two parts, depending on the real part of ω. The case $\mathrm{Re}(\omega) > -\frac{1}{2}$ is a direct consequence of Gambaudo and Ghys' results [9]. The case $\mathrm{Re}(\omega) < -\frac{1}{2}$ involves a special kind of plumbing operation on positive 3-braids. We would like to stress that our proof breaks down in the limit points $\omega = 1$ and -1. In particular, our argument does not recover Feller's result for the classical signature invariant $\sigma = \sigma_{-1}$. For a similar reason, our argument does not imply that the zeros of the Alexander polynomial of random positive braids equidistribute on S^1. This interesting problem remains open.

2. Gambaudo–Ghys and $\mathrm{Re}(\omega) > -\frac{1}{2}$

In their influential article on braids, Gambaudo and Ghys derived that the restriction of the Levine–Tristram signature function $\sigma_{\omega=e^{2\pi i\theta}}$ to every fixed braid group B_k with $k \geq 3$ is quasi-linear on the interval $\theta \in (0, \frac{1}{k})$, with the slope proportional to the linking number lk. More precisely, for all $\beta \in B_k$ and $\theta \in (0, \frac{1}{k})$ with $e^{2\pi i\theta} \notin \bar{\mathbb{Q}}$, the function $\theta \mapsto \sigma_{\omega=e^{2\pi i\theta}}(\beta)$ satisfies

$$|\sigma_{\omega=e^{2\pi i\theta}}(\beta) - 2\theta \mathrm{lk}(\beta)| \leq k - 1$$

(see Corollary 4.4. in [9] for the case $k = 3$, and the remark just before Corollary 4.4. for general $k \geq 3$). Here, we use the convention that positive braid links have positive signature, contrary to Gambaudo and Ghys. In the case of non-split positive braids $\beta \in B_k^+$, the linking number can be expressed by the first Betti number $\mathrm{lk}(\beta) = b_1(\beta) + k - 1$. As a consequence, we obtain the following estimate for all $\theta \in (\frac{1}{k+1}, \frac{1}{k})$:

$$\liminf \left\{ \frac{\sigma_{e^{2\pi i\theta}}(\beta)}{b_1(\beta)} \middle| \beta \in B_k^+ \right\} = \liminf \left\{ \frac{\sigma_{e^{2\pi i\theta}}(\beta)}{\mathrm{lk}(\beta)} \middle| \beta \in B_k^+ \right\} > \frac{2}{k+1}.$$

This settles Theorem 1.1 for all $\omega \in S^1 \setminus \bar{\mathbb{Q}}$ with $\mathrm{Re}(\omega) > \mathrm{Re}(\zeta_3) = -\frac{1}{2}$; by Feller's method described in [7] write $\omega = e^{2\pi i\theta}$ with $\theta \in (\frac{1}{k+1}, \frac{1}{k})$ and $k \geq 3$ (here, we use $\mathrm{Re}(\omega) > -\frac{1}{2}$) and let the above limit inferior be $\frac{2c}{k+1}$ with $c > 1$. Fix any braid $\alpha \in B_n^+$, with $n \geq k$. We can turn α into a finite union of positive k-braids $\bar{\alpha}$, by smoothing a ratio of at most $\frac{1}{k}$ crossings of α. Then, up to a fixed constant error:

$$\begin{aligned}
\sigma_\omega(\alpha) &\geq \sigma_\omega(\bar{\alpha}) - \frac{1}{k} b_1(\alpha) \\
&\geq \frac{2c}{k+1} b_1(\bar{\alpha}) - \frac{1}{k} b_1(\alpha) \\
&\geq \frac{2c}{k+1} \left(1 - \frac{1}{k}\right) b_1(\alpha) - \frac{1}{k} b_1(\alpha) \\
&= \frac{2c(k-1) - (k+1)}{k(k+1)} b_1(\alpha).
\end{aligned}$$