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REVERSIBLE POISSON–KIRCHHOFF SYSTEMS

by Alexandre Boyer, Jérôme Casse, Nathanaël Enriquez & Arvind Singh

Abstract. — We define a general class of random systems of horizontal and vertical weighted broken lines on the quarter plane whose distribution are proved to be translation invariant. This invariance stems from a reversibility property of the model. This class of systems generalises several classical processes of the same kind, such as Hammersley's broken line processes involved in last passage percolation theory or such as the six-vertex model for some special sets of parameters. The novelty here comes from the introduction of a weight associated with each line. The lines are initially generated by spatially homogeneous weighted Poisson point process and their evolution (turn, split, crossing) are ruled by a Markovian dynamics, which preserves Kirchhoff's node law for the line weights at each intersection. Among others, we derive some new explicit invariant measures for some bullet models, as well as new reversible properties for some six-vertex models with an external electromagnetic field.

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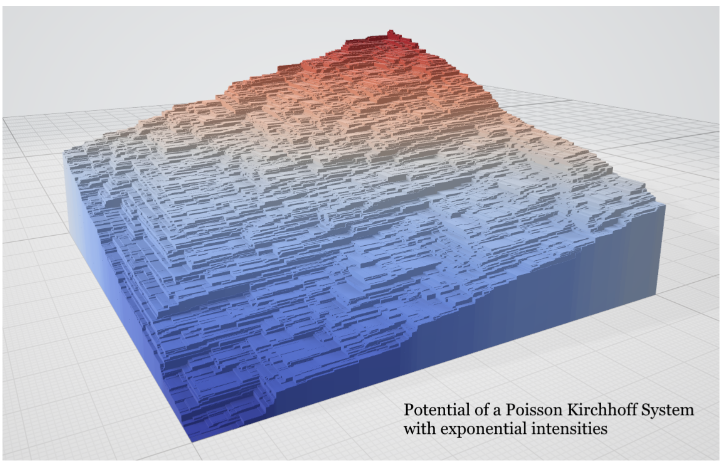
Résumé (*Systèmes de Poisson-Kirchhoff réversibles*). — Nous définissons une classe générale de systèmes aléatoires de lignes brisées pondérées, horizontales et verticales, dans le quart de plan pour lesquels nous prouvons qu'il existe des lois invariantes par translation. Cette invariance découle d'une propriété de réversibilité du modèle. Cette classe de systèmes généralise plusieurs processus classiques du même type, comme les processus de lignes brisées d'Hammersley apparaissant en percolation de dernier passage, ou bien comme le modèle à six-vertex pour des valeurs spécifiques de paramètres. La nouveauté du papier vient de l'introduction d'un poids associé à chaque ligne. Les lignes sont intialement distribuées suivant un processus poncuel de Poisson pondéré et spatialement homogène, et leur évolution (virage, division, croisement) est décrite par une dynamique markovienne pour laquelle le poids des lignes satisfait la loi des noeuds de Kirchhoff à chaque intersection. Entre autres, nous obtenons de nouvelles mesures invariantes explicites pour des modèles ballistiques ainsi que de nouvelles propriétés de réversibilité pour des modèles à six-vertex avec champ électromagnétique externe.



1. Introduction

In his seminal work [21], Hammersley introduced his now famous *broken line process* as a means to study the length of the longest increasing sequence in a random permutation. This model of last passage percolation (LPP) enjoys many remarkable properties and has since been thoroughly scrutinised [29, 30]. One possible construction of Hammersley's process on the quarter plane $[0,\infty)^2$ is as follows: consider a unit intensity Poisson point process (PPP) on $[0,\infty)^2$. Each atom of the point process "emits" a pair of particle/anti-particle with the particle of charge $+1$ moving horizontally to the right and the anti-particle with charge -1 moving upward. When the traces of two particles of opposite charge meet, they both disappear. Then, the collection of all traces obtained with this procedure is exactly Hammersley's broken line process on the quarter plane (see

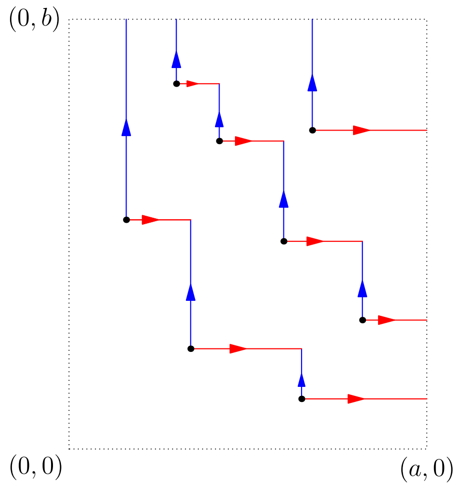


FIGURE 1.1. A realisation of Hammersley's broken line process in the rectangle $[0, a] \times [0, b]$. The traces of particles with charge $+1$ are represented in red, and those of the anti-particles with charge -1 are represented in blue.

Figure 1.1 for an illustration of the construction). Let us note that, in view of this construction, the system may be called "conservative" in the sense that the total charge of the system remains null since particles and anti-particles appear and disappear simultaneously.

In this paper, we introduce a new class of random processes, which we call *Poisson–Kirchhoff Systems* (PKS) that generalise the construction described above. Those processes again consist of random collections of weighted horizontal and vertical broken lines living on the quarter plane $[0, \infty)^2$. As for Hammersley's broken line process, one may think of these lines as being the traces of "charged" particles moving either horizontally (i.e. increasing their x-coordinate) or vertically (i.e. increasing their y-coordinate). However, in this new class of processes, particles may hold arbitrary charges and may randomly turn, split or coalesce, according to a special Markovian dynamics that is still conservative in the sense that the total charge remains constant. We show in this paper that, when the parameters of the dynamics take a particular form, the PKS process is spatially reversible. Then, it is possible to construct a translation invariant PKS process on the whole plane whose marginal distribution along vertical and horizontal lines is (weighted) PPPs.

The paper is organised as follows. In Section 2, we define the PKS process in a general setting and prove its existence under a uniform boundedness assumption on the parameters.

In Section 3, we introduce a notion of reversibility for PKS processes, which essentially says that the distribution of a PKS restricted to any rectangular box is invariant by a rotation of 180°. Then we present, in our main results, suitable conditions that guarantee the reversibility and, therefore, the invariance of PKS processes. We do it in three different frameworks according to whether the distribution of the line weights is absolutely continuous with respect to the Lebesgue measure, discrete or arbitrary.

The proof of this reversibility property is carried out in Section 5. The state space of PKS processes is quite complicated, and in order to deal with it, we introduce a family of parametrisations. It turns out that two different parametrisations of this family define the same volume form. We apply this result to two specific parametrisations: the first one associated to the dynamics of the PKS and the second one associated to its reverse dynamics. Once we have done this, a careful analysis shows that the densities associated to the dynamics and to the reversed one in their respective parametrisations coincide under the above-mentioned conditions. Interestingly, one can exploit this invariance result in order to extend the proof of the existence of the PKS to unbounded parameters.

In Section 6, we first show how Kirchhoff's node law makes it possible to define a notion of potential function associated with the faces of the tessellation defined by a PKS. This potential function corresponds to the last passage times in LPP. We then collect several LPP models that can be mapped to PKS processes. In the sequel, we provide a (non-exhaustive) list of PKS processes obtained for specific distributions of the line weights. From this list, we recover several other classical models of statistical physics like bullet models [8, 22, 24] or six-vertex models [4, 28]. In particular, we exhibit some new explicit invariant measures for some bullet models, as well as new reversible properties for some six-vertex models with an external electromagnetic field. Furthermore, the special cases of Gaussian or Poisson distributions for the line weights provide new models with explicit dynamics that may be worthy of further study.

Finally, in Section 7, we look at basic geometric properties of the random tessellation of the quarter plane induced by a PKS, such as the mean number of connected components inside a rectangle and the mean number of nodes of a typical connected component.

2. Poisson–Kirchhoff systems

The definition of a generic Poisson–Kirchhoff process relies on nine parameters. First, let λ_0 be a non-negative number, which will be referred to as the *spontaneous creation rate.* Let λ_V and λ_H be two functions from $\mathbb{R}$ to $\mathbb{R}_+$, called *vertical and horizontal split rate functions.* Let τ_V and τ_H be two functions from $\mathbb{R}$ to $\mathbb{R}_+$, called *vertical and horizontal turn rate functions.* Let $p_0 \in [0,1]$

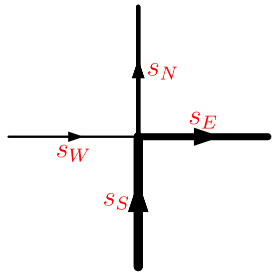


FIGURE 2.1. Kirchhoff's node law at a crossing: two lines come from the south and the west directions with respective weights s_S and s_W. Lines exiting the intersection in the north and west direction have respective weights s_N and s_E. The sum of weights entering and exiting the intersection is conserved: $s_S + s_W = s_N + s_E$.

be called the *annihilation probability.* Also let p_V and p_H be two functions from $\mathbb{R}$ to $[0, 1]$, called, respectively, *vertical* and *horizontal coalescence probability functions* that satisfy, for any $s \in \mathbb{R}$,

$$p_V(s) + p_H(s) + p_0 \mathbf{1}_{s=0} \leq 1. \tag{1}$$

Finally, let $F = (F(s, \cdot) : s \in \mathbb{R})$ be a probability transition kernel on $\mathbb{R}$, called the *division kernel*, which satisfies:

- The map $s \mapsto F(s, B)$ is $\mathcal{B}(\mathbb{R})$-measurable for any Borel set $B \in \mathcal{B}(\mathbb{R})$.
- $B \mapsto F(s, B)$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for any $s \in \mathbb{R}$.

The collection $(\lambda_0, \lambda_V, \lambda_H, p_0, p_V, p_H, \tau_V, \tau_H, F)$ represents the parameters of the model. The three parameters $(\lambda_0, \lambda_V, \lambda_H)$ can be seen as splitting rates, whereas (p_0, p_V, p_H) can be seen as merging probabilities. We will see that these two sets of parameters play a dual role. The two parameters (τ_V, τ_H) have a symmetric role and describe how often vertical and horizontal lines turn. Finally, the kernel F describes the distribution of the weights when a line splits, or when two lines meet and split again.

We now define a random system of horizontal and vertical algebraic weighted lines inside the quarter plane $[0, \infty)^2$, which preserves Kirchhoff's node law at every intersection (with respect to their weights), as prescribed in Figure 2.1. As in the description of Hammersley's process in Section 1, one can think of those lines as the traces of charged particles moving either to the right or upwards. Let us emphasise that, in our setting, the weight (i.e. charge) of a line may be positive, negative or even null.

We define the *initial condition* of our process by specifying the positions and weights of the vertical (or horizontal) lines that start from the x-axis (or y-axis). To this end, we fix two sets of weighted points: $\mathcal{C}_X$ on the positive