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CURVATURE ESTIMATE AND THE RAMIFICATION OF THE HOLOMORPHIC MAPS OVER HYPERSURFACES ON RIEMANN SURFACES

BY SI DUC QUANG

ABSTRACT. — In this paper, we give a Gauss curvature estimate on an open Riemann surface S whose metric is of the form

$$ds^2 = \|G\|^{2m}|\omega|^2,$$

where ω is a holomorphic 1-form, $m \in \mathbf{Z}_+$ and G is a reduced representation of a holomorphic map g from S into a projective subvariety V of $\mathbb{P}^n(\mathbb{C})$ that is ramified over a family of hypersurfaces in N-subgeneral position with respect to V.

RÉSUMÉ (*Estimation de la courbure et ramification des applications holomorphes avec les hypersurfaces sur des surfaces de Riemann*). — Dans cet article, nous donnons une estimation de la courbure de Gauss sur une surface de Riemann ouverte S dont la métrique est de la forme

$$ds^2 = \|G\|^{2m}|\omega|^2,$$

où ω est une 1-forme holomorphe, $m \in \mathbb{Z}_+$ et G est une représentation réduite d'une application holomorphe g de S dans une sous-variété projective V de $\mathbb{P}^n(\mathbb{C})$ qui est ramifiée sur une famille d'hypersurfaces en position N-subgénérale par rapport à V.

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1. Introduction and Main results

In 2021, X. Chen, Y. Li, Z. Liu and M. Ru proved the following theorem.

THEOREM A (X. Chen et al. [3]). — *Let S be an open Riemann surface and let $g : S \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Consider the conformal metric on S given by*

$$ds^2 = \|G\|^{2m}|\omega|^2,$$

where G is a reduced representation of g, ω is a holomorphic 1-form, and $m \in \mathbf{Z}_+$. Assume that ds^2 is complete. If g is k-nondegenerate for some k with $1 \le k \le n$, then g can omit at most $mk\left(n - \frac{k-1}{2}\right) + 2n - k + 1$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position.

Without the assumption that the metric ds^2 is complete, these four authors proved the following estimate of the Gauss curvature.

THEOREM B (X. Chen et al. [3, Theorem 2]). — *Let S be an open Riemann surface and let $g : S \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Consider the conformal metric on S given by*

$$ds^2 = \|G\|^{2m}|\omega|^2,$$

where G is a reduced representation of g, ω is a holomorphic 1-form, and $m \in \mathbf{Z}_+$. Assume that g omits more than $\frac{n+1}{2}(mn+2)$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position. Then there exists a constant C depending only on the set of omitted hyperplanes such that

$$|K(p)|^{\frac{1}{2}} d(p) \le C,$$

where $K(p)$ is the Gauss curvature of S at p with respect to ds^2, and $d(p)$ is the geodesic distance from p to the boundary of S.

Our aim in this paper is to generalize the above results to the cases of the holomorphic map g and a family of hypersurfaces $\{Q_j\}_{j=1}^q$. Moreover, we will replace the condition that "*g omits hyperplanes $\{H_j\}_{j=1}^q$*" by a weaker condition that "*g is ramified over Q_j with multiplicity at least m_j*". In order to do so, firstly we have to establish the general second main theorem with truncation multiplicity for holomorphic maps into projective varieties and hypersurfaces in subgeneral position, and then show a normality criterion for the families of holomorphic maps on a complex domain that are ramified over some hypersurfaces in general position of $\mathbb{P}^n(\mathbb{C})$. Secondly, we will construct a pseudo-metric with negative curvature on Riemann surfaces induced by a holomorphic curve, which ramified some hypersurfaces with multiplicity at least a certain level. To state our results, we introduce the following notions.

Let V be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension k $(k \le n)$. Let d be a positive integer. We denote by $I(V)$ the ideal of homogeneous polynomials in $\mathbb{C}[x_0,\dots,x_n]$ defining V and by $\mathbb{C}[x_0,\dots,x_n]_d$ the vector space of all homogeneous polynomials in $\mathbb{C}[x_0,\dots,x_n]$ of degree d including the zero polynomial. Define

$$I_d(V) := \frac{\mathbb{C}[x_0,\dots,x_n]_d}{I(V)\cap\mathbb{C}[x_0,\dots,x_n]_d} \text{ and } H_V(d) := \dim I_d(V).$$

Then $H_V(d)$ is called the Hilbert function of V. Each element of $I_d(V)$ that is an equivalent class of an element $Q \in \mathbb{C}[x_0,\dots,x_n]_d$, will be denoted by $[Q]$.

Let $f : S \longrightarrow V$ be a holomorphic map. We say that f is degenerate over $I_d(V)$ if there is a $[Q] \in I_d(V)\setminus\{0\}$ such that $Q(F)\equiv 0$ for some local reduced representation F of f. Otherwise, we say that f is nondegenerate over $I_d(V)$.

Let m be a positive integer or $m = +\infty$. The holomorphic map f is said to be ramified over a hypersurfaces Q with multiplicity m if:

(1) *either* the image of f is contained in Q,
(2) *or* $\nu_{Q(f)}(z) \geq m$ for every $z \in \operatorname{Supp}\nu_{Q(f)}$, where $\nu_{Q(f)}$ is the divisor f^*Q.

Let $Q_1,\dots,Q_q$ $(q \geq k+1)$ be q hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. The family of hypersurfaces $\{Q_i\}_{i=1}^q$ is said to be in N-subgeneral position with respect to V if

$$V \cap \left(\bigcap_{j=1}^{N+1} Q_{i_j}\right) = \varnothing \ \forall\ 1 \leq i_1 < \cdots < i_{N+1} \leq q.$$

Our first main result is stated as follows.

Theorem 1.1. — *Let S be an open Riemann surface. Let V be a projective subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $Q_1,\dots,Q_q$ be hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in N-subgeneral position with respect to V. Let d be the least common multiple of $\deg Q_j$ $(1 \leq j \leq q)$, i.e., $d = \operatorname{lcm}(\deg Q_1,\dots,\deg Q_q)$. Let $g : S \to V \subset \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Consider the conformal metric on S given by*

$$ds^2 = \|G\|^{2m}|\omega|^2,$$

where G is a reduced representation of g, ω is a holomorphic 1-form, and $m \in \mathbf{Z}_{\geq 0}$. Assume that ds^2 is complete, g is nondegenerate over $I_d(V)$ and g is ramified over each Q_j with multiplicity at least m_j $(1 \leq j \leq q)$. Then we have

$$\sum_{j=1}^{q}\left(1-\frac{M}{m_j}\right) \leq \frac{(2N-k+1)(M+1)}{k+1} + \frac{m(2N-k+1)M(M+1)}{2d(k+1)},$$

where $M = H_V(d)-1$ and $k = \min\operatorname{rank}_{\mathbb{C}}\{[Q_{i_0}],\dots,[Q_{i_N}]; 1 \leq i_0 < \cdots < i_N \leq q\} - 1$,

Here, we may see that $M \leq \binom{n+d}{n} - 1$ and $\dim V \leq k \leq \min\{M, N\}$. From Theorem 1.1, we have the following corollary.