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TOWARDS TEMPERED ANABELIAN BEHAVIOUR OF BERKOVICH ANNULI

by Sylvain Gaulhiac

Abstract. — This work brings to light some partial *anabelian behaviours* of analytic annuli in the context of Berkovich geometry. More specifically, if k is a valued non-archimedean complete field of mixed characteristic that is algebraically closed, and $\mathcal{C}_1$, $\mathcal{C}_2$ are two k-analytic annuli with isomorphic tempered fundamental group, we show that the lengths of $\mathcal{C}_1$ and $\mathcal{C}_2$ cannot be too far from each other. When they are finite, we show that the absolute value of their difference is bounded above with a bound depending only on the residual characteristic p.

Résumé (*Vers un comportement anabélien tempéré des couronnes de Berkovich*). — Ce travail met en lumière, partiellement, un comportement anabélien des couronnes dans le cadre de la géométrie analytique de Berkovich. Plus précisément, si k est un corps non-archimédien complet algébriquement clos de caractéristique mixte, et $\mathcal{C}_1$, $\mathcal{C}_2$ deux couronnes k-analytiques ayant des groupes fondamentaux tempérés isomorphes, nous montrons que les longueurs de ces deux couronnes ne peuvent être trop éloignées l'une de l'autre. Quand ces longueurs sont finies, nous prouvons que la valeur absolue de leur différence est bornée par une expression ne dépendant que de la caractéristique résiduelle p.

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1. Introduction

Anabelian geometry is concerned with the following question.

> *To what extent is a geometric object determined by its fundamental group?*

It is within the framework of algebraic geometry that Grothendieck gave the first conjectures of anabelian geometry in a famous letter to Faltings in 1983, where the fundamental group is nothing other than the étale one. Some deep results for hyperbolic curves have been obtained by Tamagawa and Mochizuki, answering certain conjectures of Grothendieck. However, almost no results are known for higher dimensions.

In the context of Berkovich analytic geometry, it is possible to define several "fundamental groups" classifying, for instance, *topological*, *finite étale* or *étale* (in the sense of [5]) coverings. However, the group that seems to best capture anabelian behaviours of analytic spaces over non-archimedian fields is the *tempered fundamental group*, introduced by Yves André in [2]. This group classifies *tempered coverings*, defined as étale coverings that become topological after a finite étale base change. Both finite étale and topological coverings are examples of tempered coverings.

In [1], Yves André obtained for the first time some results of anabelian nature related to the tempered fundamental group. A few years later, a huge step was taken in this direction with some results of Shinichi Mochizuki ([14]) followed by Emmanuel Lepage ([9] and [10]). These results relate the fundamental tempered group of the analytification of an algebraic hyperbolic curve to the dual graph of its stable reduction. If X is a hyperbolic curve defined over some non-archimedian complete field k, the homotopy type of its analytification X^{an} can be described in terms of the stable model $\mathscr{X}$ of X. More precisely, if $\mathscr{X}_s$ stands for the special fibre of $\mathscr{X}$, the *dual graph of the stable reduction* of X, denoted $\mathbb{G}_X$, is the finite graph whose vertices are the irreducible components of $\mathscr{X}_s$, and whose edges correspond to the nodes (singularities in ordinary double points) between irreducible components. If $\overline{X}$ denotes the normal compactification of X, a cusp of X is an element of $\overline{X} \setminus X$. Let us denote by $\mathbb{G}_X^{\mathfrak{c}}$ the graph obtained from $\mathbb{G}_X$, adding one open edge to each cusp of X, called the *extended dual graph of the stable reduction* of X. There exists a canonical topological embedding $\mathbb{G}_X^{\mathfrak{c}} \hookrightarrow X^{\mathrm{an}}$, which admits a topologically proper deformation retraction $X^{\mathrm{an}} \twoheadrightarrow \mathbb{G}_X^{\mathfrak{c}}$, and, thus, X^{an} and $\mathbb{G}_X^{\mathfrak{c}}$ have the same homotopy type.

Using the language of *semi-graphs of anabelioids* and *temperoids* introduced in high generality in [13] and [14], Mochizuki proves in [14] that the fundamental tempered group of the analytification of a hyperbolic curve determines the dual graph of its stable reduction:

THEOREM 1.1 ([14], Corollary 3.11). — *Let X_1 and X_2 be two hyperbolic curves over $\mathbb{C}_p$. Any outer isomorphism of groups $\varphi : \pi_1^{\mathrm{temp}}(X_1^{\mathrm{an}}) \xrightarrow{\sim} \pi_1^{\mathrm{temp}}(X_2^{\mathrm{an}})$ determines, functorially in φ, a unique isomorphism of graphs: $\overline{\varphi} : \mathbb{G}^{\mathrm{c}}_{X_1} \xrightarrow{\sim} \mathbb{G}^{\mathrm{c}}_{X_2}$.*

Mochizuki shows more precisely that it is possible to reconstruct the graph of the stable reduction $\mathbb{G}_X$ of a hyperbolic curve X from a (p')-version $\pi_1^{\mathrm{temp},\,(p')}(X^{\mathrm{an}})$ of the tempered fundamental group.

A few years later, Emmanuel Lepage refined this result. He proved that the knowledge of the tempered fundamental group of the analytification of a hyperbolic curve X enables us to not only reconstruct the graph $\mathbb{G}_X$, but also, in some cases, its canonical metric. This metric is such that the length of an edge corresponding to a node is the width of the annulus corresponding to the generic fibre of the formal completion on this node. It is, however, necessary to restrict ourselves to *Mumford curves*, which are defined as proper algebraic curves X over $\mathbb{C}_p$, such that the normalized irreducible components of the stable reduction are isomorphic to $\mathbb{P}^1$. This is equivalent to saying in Berkovich language that the analytification X^{an} is locally isomorphic to open subsets of $\mathbb{P}^{1,\mathrm{an}}$, or that X^{an} does not contains any point of genus > 0.

THEOREM 1.2 ([10]). — *Let X_1 and X_2 be two hyperbolic Mumford curves over $\mathbb{C}_p$ and $\varphi : \pi_1^{\mathrm{temp}}(X_1^{\mathrm{an}}) \xrightarrow{\sim} \pi_1^{\mathrm{temp}}(X_2^{\mathrm{an}})$ an isomorphism of groups. Then the isomorphism of graphs $\overline{\varphi} : \mathbb{G}_{X_1} \xrightarrow{\sim} \mathbb{G}_{X_2}$ is an isomorphism of metric graphs.*

These two results deal with analytic curves that are of *algebraic nature*, that is, analytifications of algebraic curves. Yet the theory of Berkovich analytic spaces is rich enough to contain many curves that are of *analytic nature* without coming from algebraic curves. The most important examples of such curves, which are still very simple to define, are *disks* and *annuli*. In the wake of Mochizuki's and Lepage's results, one wonders whether similar anabelian results exist for more general analytic curves without imposing any algebraic nature. The generalisation of Mochizuki's results for such analytic curves was carried out in the article [7], whereas the investigation about some analogue of Lepage's result is partially answered in this present article.

Reconstruction of the analytic skeleton. For a quasi-smooth analytic curve X, the good analogue of the extended dual graph of the stable reduction is the *analytic skeleton* $S^{\mathrm{an}}(X)$, defined in 2.5. When the skeleton meets all the connected components of X, there exists a canonical topological embedding $S^{\mathrm{an}}(X) \hookrightarrow X$, which admits a topologically proper deformation retraction $X \twoheadrightarrow S^{\mathrm{an}}(X)$. Therefore, X and $S^{\mathrm{an}}(X)$ have the same homotopy type. The restriction $S^{\mathrm{an}}(X)^{\natural}$ obtained from the skeleton by removing non-relatively compact edges is called the *truncated skeleton* of X (see 2.8), and is the analogue of the dual graph of the stable reduction. Let k be a complete algebraically closed non-archimedean field of residual exponent p. In [7], 3.29, a certain of class of

k-analytic curves is defined, called *k-analytically hyperbolic.* Their interest lies in the fact that for a k-analytically hyperbolic curve X it is possible to reconstruct its truncated skeleton $S^{\mathrm{an}}(X)^{\natural}$ from the tempered group $\pi_1^{\mathrm{temp}}(X)$, or even from a *prime-to-p* version $\pi_1^{\mathrm{temp},\,(p')}(X)$, obtained by taking the projective limit of all quotients of $\pi_1^{\mathrm{temp}}(X)$ admitting a normal torsion-free subgroup of finite index prime to p. The reconstruction of $S^{\mathrm{an}}(X)^{\natural}$ from this group is given by the following:

- the vertices correspond to the conjugacy classes of maximal compact subgroups of $\pi_1^{\mathrm{temp},\,(p')}(X)$;
- the edges correspond to the conjugacy classes of non-trivial intersections of two maximal compact subgroups of $\pi_1^{\mathrm{temp},\,(p')}(X)$.

The condition for a quasi-smooth k-analytic curve to be analytically hyperbolic is stated in terms of non-emptiness of the sets of nodes of the skeleton and some combinatorial hyperbolic condition at each of these nodes. However, the analytical hyperbolicity may not be enough to recover all the skeleton. In order to recover also the non-relatively compact edges of $S^{\mathrm{an}}(X)$ is defined in [7], 3.55, a sub-class of k-analytically hyperbolic curves called *k-analytically anabelian.* A k-analytically anabelian curve is a k-analytically hyperbolic curve satisfying a technical condition called *ascendance vicinale*, which enables us to reconstruct open edges of the skeleton:

THEOREM 1.3 ([7], 3.56). — *Let X_1 and X_2 be two k-analytically anabelian curves. Any group isomorphism $\varphi : \pi_1^{\mathrm{temp}}(X_1) \xrightarrow{\sim} \pi_1^{\mathrm{temp}}(X_2)$ induces (functorially in φ) an isomorphism of semi-graphs between the analytic skeletons: $S^{\mathrm{an}}(X_1) \xrightarrow{\sim} S^{\mathrm{an}}(X_2)$.*

Anabelianity of length? This present article concentrates more on the potential anabelianity of lengths of edges of the skeleton of a k-analytic curve, inspired from the result of Lepage cited above. There is a natural way to define the length of an analytic annulus (see 2.19), invariant by automorphisms, which makes the skeleton $S^{\mathrm{an}}(X)$ of a quasi-smooth k-analytic curve X a *metric* graph. The question that naturally arises is the following:

> *Does the tempered fundamental group $\pi_1^{\mathrm{temp}}(X)$ of a k-analytically anabelian curve X determine $S^{\mathrm{an}}(X)$ as a metric graph?*

Before tackling the general case, it seems *a priori* simpler to study first the case of a k-analytic annulus, even if this latter is not a k-analytically anabelian curve. The (p')-tempered group $\pi_1^{\mathrm{temp},\,(p')}(\mathcal{C})$ of an annulus is always isomorphic to the p'-profinite completion $\widehat{\mathbb{Z}}^{(p')}$ of $\mathbb{Z}$, but its total tempered group $\pi_1^{\mathrm{temp}}(\mathcal{C})$ depends on its length whenever k has mixed characteristic. The new question arising is the following:

> *Does the tempered group $\pi_1^{\mathrm{temp}}(\mathcal{C})$ of a k-analytic annulus $\mathcal{C}$ determine its length?*

In order to investigate this question, one is tempted to follow the scheme of proof that Lepage develops in [10]. An idea would be to start from an "ovoid" μ_p-covering of the annulus totally split at the middle of the skeleton, which would be analytically anabelian. Then knowing how to compute the length of any cycle would be enough to know the length of the annulus (by a limit argument). Yet one quickly faces problems of analytic nature that do not appear with Mumford curves: problems of detection of μ_{p^h}-torsors with trivial $\mathbb{Z}/p^h\mathbb{Z}$-cochain. Indeed, if $Y \to X$ is a μ_n-torsor, associating to some edge e of $S^{\mathrm{an}}(X)$, the growth rate of any analytic function defining locally this torsor over e leads to a harmonic cochain on the graph $S^{\mathrm{an}}(X)$ with values in $\mathbb{Z}/n\mathbb{Z}$. This growth rate corresponds to the degree of the strictly dominant monomial (see remark 2.13) of the corresponding analytic function. Therefore, when X is a quasi-smooth k-analytic curve, we show in Lemma 3.4 that there exists a cochain morphism $\theta : H^1(X, \mu_n) \to \mathrm{Harm}(S^{\mathrm{an}}(X), \mathbb{Z}/n\mathbb{Z})$, for any $n \in \mathbb{N}^\times$. However, when $n = p^h$ with $h > 1$, it seems difficult to detect the kernel of θ from $\pi_1^{\mathrm{temp}}(X)$, which makes the hoped for scheme of proof illusory. Nevertheless, the detection of $\ker(\theta)$ when $n = p$ is possible in some cases.

Theorem 0. — *Let X be a k-analytic curve satisfying one of the two following conditions:*

1. *X is an annulus;*
2. *X is a k-analytically hyperbolic curve of finite skeleton without a bridge, without boundary or any point of genus > 0, with only annular cusps and at least a finite-annular one, such that there is never strictly more than one cusp coming from each node.*

Then the set of μ_p-torsors of X with trivial $\mathbb{Z}/p\mathbb{Z}$-cochain, $H^1(X, \mu_p) \cap \ker(\theta)$, is completely determined by $\pi_1^{\mathrm{temp}}(X)$.

This result uses *resolution of non-singularities* (section 4) coupled with a characterisation of non-triviality of cochains in terms of minimality of splitting radius at rigid points (Proposition 3.8). This characterisation can be re-phrased set-theoretically with the splitting sets of torsors (corollary 3.10), which can themselves be characterised from the tempered group by means of solvability (Proposition 4.7).

As for the initial question about the potential anabelianity of lengths of annuli, we found a partial answer, using the solvability of annuli (Proposition 4.6) doubled with some considerations of splitting sets of μ_p-torsors.

Theorem 1. — *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two k-analytic annuli whose tempered fundamental groups $\pi_1^{\mathrm{temp}}(\mathcal{C}_1)$ and $\pi_1^{\mathrm{temp}}(\mathcal{C}_2)$ are isomorphic. Then $\mathcal{C}_1$ has finite length if and only if $\mathcal{C}_2$ has finite length. In this case:*

$$|\ell(\mathcal{C}_1) - \ell(\mathcal{C}_2)| < \frac{2p}{p-1}.$$