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SIMPLICIAL CHERN–WEIL THEORY FOR COHERENT ANALYTIC SHEAVES, PART I

BY TIMOTHY HOSGOOD

ABSTRACT. — In "Chern classes for coherent sheaves", H.I. Green constructs Chern classes in de Rham cohomology of coherent analytic sheaves. We construct here a formal $(\infty,1)$-categorical framework into which we can place Green's work and generalise it, also obtaining a better idea as to what exactly a *simplicial connection* should be. The result will be the ability to work with generalised invariant polynomials (which will be introduced in the sequel to this paper) evaluated at the curvature of so-called *admissible* simplicial connections to get explicit Čech representatives in de Rham cohomology of characteristic classes of coherent analytic sheaves.

RÉSUMÉ (*La théorie de Chern–Weil simpliciale pour les faisceaux analytiques cohérents, partie I*). — Dans la thèse « Chern classes for coherent sheaves », H.I. Green construit des classes de Chern dans la cohomologie de de Rham pour les faisceaux analytiques cohérents. On construit ici un cadre $(\infty,1)$-catégorique formel dans lequel on peut obtenir une meilleure idée de ce que devrait être une connexion simpliciale en généralisant le travail de Green. Il en résulte la possibilité de travailler avec les polynômes invariants généralisés (qui seront introduits dans la suite de cet article) évalués en la courbure de connexions simpliciales dites admissibles afin d'obtenir les représentants de Čech explicites dans la cohomologie de de Rham des classes caractéristiques pour les faisceaux analytiques cohérents.

This paper is one of two to have been extracted from the author's PhD thesis [15]. Further details and a more lengthy exposition can be found there.

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1. Introduction

1.1. History and motivation. — In 1980, H.I. Green, a student of O'Brian and Eells, wrote their thesis [11] on the subject of Chern classes of coherent sheaves on complex-analytic manifolds. Although the thesis was never published, an exposition was given in [26], alongside a sketch of a proof of the Hirzebruch–Riemann–Roch formula for this construction of Chern classes. It combined the theory of twisting cochains, used with great success by Toledo, Tong, and O'Brian in multiple papers ([24, 25, 20, 21]), along with the fibre integration of Dupont ([9]), to construct, from a coherent analytic sheaf, classes in $\mathrm{H}^{2k}(X, \Omega_X^{\bullet\geqslant k})$ that coincide with those given by the classical construction of Chern classes in $\mathrm{H}^{2k}(X, \mathbb{Z})$ by Atiyah–Hirzebruch ([1]). (It is of historical interest to mention that the approach of expressing characteristic classes in terms of transitions functions is very much in line with ideas propounded by Bott; see e.g. the subsection entitled 'Concluding Remarks' in [8, §23].) This construction was considered by Grivaux in his thesis [12], where he constructs unified Chern classes for coherent analytic sheaves (on *compact* analytic manifolds) in Deligne cohomology, and where he states an axiomatisation of Chern classes that ensures uniqueness in any sufficiently nice cohomology theory (of which de Rham cohomology, as well as truncated de Rham cohomology, is an example). Although he states that the Grothendieck–Riemann–Roch theorem for closed immersions is not known for Green's construction of Chern classes if X is non-Kähler, this turns out to not be a problem, since it follows from his other axioms by a purely formal, classical argument, involving deformation to the normal cone.

One reason that the study of Chern classes of coherent *analytic* sheaves is interesting is that the techniques used in the algebraic setting seem to be entirely inapplicable to the analytic setting. In both the analytic and algebraic settings, Chern classes of locally free sheaves can be constructed by the splitting principle (as explained in, e.g. [8, §21]) in the "most general" cohomology theories (Deligne–Beilinson cohomology and Chow rings, respectively); but, although coherent algebraic sheaves admit global locally free resolutions, the same is *not* true of coherent analytic sheaves. In general, complex manifolds have very few holomorphic vector bundles, and there are whole classes of examples of coherent sheaves that do not admit a global locally free resolution ([28, Corollary A.5]). One key insight of [11], however, was that the holomorphic twisting resolutions of Toledo and Tong (whose existence was guaranteed by

[25, Proposition 2.4]) could be used to construct a global resolution by "*simplicial* locally free sheaves", or **locally free sheaves on the nerve**: objects that live over the Čech nerve $X_\bullet^{\mathcal{U}}$ of a cover $\mathcal{U}$ of X. The existence of such a global resolution, glued together from local pieces, is mentioned in the introduction of [14] as a problem that should be amenable to the formal theory of descent. Indeed, these "simplicial sheaves" can be constructed by taking the lax homotopy limit (in the sense of [5, Definition 3.1]) of the diagram of model categories given by the pullback–pushforward Quillen adjunctions along the nerve of a cover of X. One very useful example of such an object is found by pulling back a global (i.e. classical) vector bundle to the nerve: given some $E \twoheadrightarrow X$, defining $\mathcal{E}^\bullet$ by $\mathcal{E}^p = (X_p^{\mathcal{U}} \to X)^* E$. This actually satisfies a "strongly Cartesian" property; it is given by the "strict" (i.e. not lax) homotopy limit of [5]. In hopefully self-explanatory notation,

$$\operatorname*{laxholim}_{[p]\in\Delta} \mathsf{Sh}(X_p^{\mathcal{U}}) \simeq \mathsf{Sh}(X_\bullet^{\mathcal{U}})$$

$$\operatorname*{holim}_{[p]\in\Delta} \mathsf{Sh}(X_p^{\mathcal{U}}) \simeq \mathsf{Sh}^{\mathrm{cart}}(X_\bullet^{\mathcal{U}}).$$

The twisting cochains from which Green builds these resolutions are also interesting objects in their own right, having been studied extensively by Toledo, Tong, and O'Brian, as mentioned previously. In fact, they can be seen as specific examples of the twisted complexes of [7], which are used to pretriangulate arbitrary dg-categories. This gives a possible moral (yet entirely informal) reason to expect the existence of resolutions such as Green's: twisted complexes give the "smallest" way of introducing a stable structure on a dg-category, and perfect $\mathcal{O}_X$-modules can be defined as exactly the objects of the "smallest" stable $(\infty,1)$-category that contains $\mathcal{O}_X$ and is closed under retracts. Alternatively, one can appeal to [29], which shows that, under certain restrictions on $(X,\mathcal{O}_X)$, twisting cochains constitute a dg-enrichment of the derived category of perfect complexes. We also mention here one more fact, shown in [6, Corollary 3 and Proposition 11] in the language of dg-categories: analogously to how sheaves (or Cartesian sheaves) on the nerve can be recovered as a lax (or not lax) homotopy limit of sheaves on each simplicial level, we can recover twisting cochains (or perfect twisting cochains) as a homotopy limit of locally free sheaves (or perfect complexes) on each simplicial level, i.e.

$$\operatorname*{holim}_{[p]\in\Delta} \mathsf{LocFree}(X_p^{\mathcal{U}}) \simeq \mathsf{Tw}(X)$$

$$\operatorname*{holim}_{[p]\in\Delta} \mathsf{Perf}(X_p^{\mathcal{U}}) \simeq \mathsf{Tw}_{\mathsf{perf}}(X).$$

Another problem in trying to apply Chern–Weil theory to coherent analytic sheaves is that global holomorphic (Koszul) connections rarely exist; the Atiyah class — which coincides with the first Chern class in cohomology — measures the obstruction of the existence of such a connection. The other main result of