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ON THE SIMILARITY BETWEEN THE IWASAWA PROJECTION AND THE DIAGONAL PART

by

J.J. Duistermaat

1. Statement of the result.

Let G be a real connected semisimple Lie group with finite center and G = KAN its Iwasawa decomposition. Via the adjoint representation, and with respect to a suitable basis in $\mathfrak{g}$, K, resp. A, resp. N are the set of matrices in G which are orthogonal, resp. diagonal with positive entries, resp. upper triangular.

The Iwasawa projection H from G onto the Lie algebra $\mathfrak{a}$ of A is defined by

(1.1) $\qquad x \in K.\exp H(x).N, \quad x \in G.$

Obviously H factorizes through the projection from G onto the (non-compact Riemannian) symmetric space K\G. If $\mathfrak{s}$ (called $\mathfrak{p}$ by everybody else) denotes the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the killing form, then the Cartan decomposition G = K.exp $\mathfrak{s}$ yields that

(1.2) $\qquad \mathfrak{s} \overset{\exp}{\to} G \to K\backslash G$

is a diffeomorphism from $\mathfrak{s}$ onto K\G. So the Iwasawa projection can be studied by looking at the mapping

(1.3) $\qquad \gamma = H \circ \exp : \mathfrak{s} \to \mathfrak{a}.$

On the other hand we have the orthogonal projection

$$\pi : \mathfrak{s} \to \mathfrak{a} \tag{1.4}$$

with respect to the Killing form. In the above matrix terminology, $\mathfrak{s}$ is the space of symmetric matrices in $\mathfrak{g}$ and π is the operation of taking the diagonal part of the symmetric matrix. So this projection has a very simple minded interpretation, whereas the Iwasawa projection is a rather more mysterious object.

<u>Theorem</u> 1.1. There is a real analytic map $\Psi : \mathfrak{s} \to K$ such that

i) $\Phi_X : k \to k.\Psi(\mathrm{Ad}\, k^{-1}(X))$ is a diffeomorphism from K onto K, for each $X \in \mathfrak{s}$.

ii) $\gamma(\mathrm{Ad}\, \Psi(X)^{-1}(X)) = \pi(X)$ for all $X \in \mathfrak{s}$.

That is, we can turn the Iwasawa projection into the orthogonal projection by an action of Ad K, the element of K depending analytically on $X \in \mathfrak{s}$.

It also follows from the theorem that the images of an Ad K-orbit in $\mathfrak{s}$ under γ and π are the same. This was obtained before by Kostant [4] who showed separately that both images are equal to the convex hull of the intersection of the Ad K-orbit in $\mathfrak{s}$ with $\mathfrak{a}$. Since this intersection is equal to a Weyl group orbit in $\mathfrak{a}$, which is finite, this image is a convex polytope. Very remarkable because an Ad K-orbit is such a roundish object!

Later Heckman [3] reduced the convexity theorem for the Iwasawa projection to the convexity theorem for the diagonal part, for which the proof is much simpler, using a homotopy argument. This homotopy argument actually is one of the elements in the proof of Theorem 1.1.

For me the major motivation for wanting the theorem was the study in [2], together with Kolk and Varadarajan, of the asymptotic behaviour of integrals of the form

$$I_a(X,\xi) = \int_K e^{i\langle\gamma(\mathrm{Ad}\, k^{-1}(X)),\xi\rangle}.a(X,k)dk \tag{1.5}$$

as $\|\xi\| \to \infty$, $\xi \in \mathfrak{a}^*$. The matrix coefficients of the principal series representations of G are given by such integrals, the simplest case being the elementary spherical functions where

$$a(X,k) = e^{-\langle\gamma(\mathrm{Ad}\, k^{-1}(X)),\rho\rangle}. \tag{1.6}$$

The idea in [2] was to consider (1.5) as an oscillatory integral, for which the asymptotics is concentrated at the stationary points of the "phase function".

$$F_{X,\xi} : k \to \langle\gamma(\mathrm{Ad}\, k^{-1}(X)),\xi\rangle \tag{1.7}$$

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on K. We then observed that $F_{X,\xi}$ had exactly the same critical points and critical values as its "infinitesimal counterpart"

$$(1.8)\qquad f_{X,\xi} = \lim_{t\to 0} \frac{1}{t} F_{tX,\xi} : k \to \langle\pi(\mathrm{Ad}\, k^{-1}(X)),\xi\rangle.$$

These critical points in turn had such a special, rigid structure that the asymptotics of (1.5) could be obtained by a repeated application of the method of stationary phase.

It had already been observed in [2] that the equality of critical points and critical values of $F_{X,\xi}$ and $f_{X,\xi}$ leads to the existence of a diffeomorphism $\Phi_{X,\xi} : K \to K$ such that $F_{X,\xi}\circ\Phi_{X,\xi} = f_{X,\xi}$.

However, the diffeomorphism is not unique and at that time I could not find $\Phi_{X,\xi}$ depending smoothly on X and ξ. Already continuous dependence on ξ would imply, replacing ξ by $t\xi$, dividing by t, and letting $t \to 0$, that $F_{X,\xi}\circ\Phi_{X,0} = f_{X,\xi}$. That is, one could find a diffeomorphism Φ_X not depending on ξ. Then, using the substitution of variables

$$(1.9)\qquad k = \Phi_X(l), \quad l \in K,$$

the integral (1.5) can be rewritten as ($X \in \mathfrak{s}$, $\xi \in \mathfrak{a}^*$)

$$(1.10)\qquad I_a(X,\xi) = \int_K e^{i\langle\pi(\mathrm{Ad}\, k^{-1}(X)),\xi\rangle} a(X,\Phi_X(k)).\left|\det \frac{\partial\Phi_X}{\partial k}(k)\right| dk.$$

In this way the study of the asymptotic behaviour would be reduced to doing stationary phase with the simpler $f_{X,\xi}$ as the phase function, rather than $F_{X,\xi}$. (Such asymptotics has been done before by Clerc and Barlet [1].)

It is one of the applications of Theorem 1.1, that the integral representation (1.10) actually holds with a $\Phi_X(k)$ which depends analytically on X and k simultaneously. For instance, for the elementary spherical functions this leads to an integral formula of the form

$$(1.11)\qquad \phi_\xi(\exp X) = \int_K e^{i\langle\pi(\mathrm{Ad}\, k^{-1}(X)),\xi\rangle} b(\mathrm{Ad}\, k^{-1}(X))dk,$$

for some analytic function $b : \mathfrak{s} \to \mathbb{R}$. As an application of the analyticity of b, one can note that replacing ξ, resp. X by $i\xi$, resp. iX, one obtains the elementary spherical functions for the *compact* symmetric space which is dual to $K\backslash G$. (In this case ξ has to be taken in a weight lattice.) So also for these functions an integral formula like (1.10) holds, at least for small $\|X\|$. I owe this observation to Richard van den Dries (T.H. Delft), who is using this integral formula in his characterization of invariant pseudo-differential operators on compact symmetric spaces in terms of their eigenvalues.

2. SL(2,ℝ).

For $G = SL(2,\mathbb{R})$, $\dim K = \dim \mathfrak{a}$ (= 1), so the substitution of variables is unique up to a flip. In order to determine it explicitly, write the elements of K as

$$k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \theta \in \mathbb{R}/2\pi\mathbb{Z}, \tag{2.1}$$

and the elements of $\mathfrak{a}$ as

$$X = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \quad t \in \mathbb{R}. \tag{2.2}$$

Then $Y = \mathrm{Ad}\, k^{-1}(X) = k^{-1}Xk$ is the general element of $\mathfrak{s}$, and $\Phi_X(k)$ is the element of K with the coordinate μ given implicitly by

$$e^{2t}\cos^2\mu + e^{-2t}\sin^2\mu = e^{2t\cos 2\theta}. \tag{2.3}$$

From this one can determine $\Psi(Y) = k^{-1}.\Phi_X(k)$. It is not entirely trivial to verify that this defines a real analytic mapping $\Psi : \mathfrak{s} \to \mathfrak{a}$!

The Jacobian of Φ_X is equal to

$$\frac{2|t\sin 2\theta|.e^{t\cos 2\theta}}{\sqrt{2}\sqrt{\cosh(2t) - \cosh(2t\cos 2\theta)}}, \tag{2.4}$$

leading to the following formula for the elementary spherical function:

$$\phi_\xi(\exp X) = \frac{1}{2\pi} \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{it\tau\cos 2\theta} \frac{|t\sin 2\theta|}{\sqrt{\frac{\cosh(2t) - \cosh(2t\cos 2\theta)}{2}}}\, d\theta. \tag{2.5}$$

Here we have written $\langle X,\xi\rangle = t\tau$. This can also be written as

$$\phi_\xi(\exp X) = \frac{4}{\pi} \int_0^t \cos\tau s . \frac{ds}{\sqrt{\frac{1}{2}(\cosh(2t) - \cosh(2s))}}. \tag{2.6}$$

A similar formula for all rank one symmetric spaces can be found in Koornwinder [8], formula (2.16) and (2.18).

I prefer (2.5) over (2.6), because there are no boundary points nor singularities for the integrand as in (2.6). To see the analyticity of the integrand in (2.5) we write