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CONTINUATION OF UNITARY DERIVED FUNCTOR MODULES OUT OF THE CANONICAL CHAMBER

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Résumé

On décrit une méthode qui permet de suivre l'unitarité lors de la continuation cohérente des représentations des séries discretes quand le parametre sort de la chambre de Weyl de Borel-de Siebenthal. Au cas où les modules de représentations des séries discretes sont obtenus en appliquant le foncteur dérivé de Zuckerman à un module de Verma généralisé qui est construit à partir d'une représentation unidimensionelle, la methode est décrite explicitement. Des programmes d'ordinateurs traitant les cas E_6, E_7, E_8 sont présentés en appendix. Un certain nombre de nouvelles représensations singulières unitaires résultent de cette méthode.

Abstract

A method is described for following unitarity during coherent continuation of discrete series representations as the parameter passes out of the Borel-de Siebenthal Weyl chamber. In the case where the discrete series representations are derived functor modules, obtained from generalized Verma modules which in turn are induced from one-dimensional representations, the computation is carried out explicitly, and computer programs are appended which treat the E_6, E_7 and E_8 cases. A number of new singular unitary representations are produced.

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Let G be a real semisimple Lie group, let θ be a Cartan involution of G, and let K denote the maximal compactly embedded subgroup of G which is the fixed point set of θ. We write $\mathfrak{g}_0, \mathfrak{k}_0$ for the real Lie algebras, $\mathfrak{g}, \mathfrak{k}$ for their complexifications, and θ for the automorphisms induced on $\mathfrak{g}_0$ and $\mathfrak{g}$. In the work of Wallach [7] and Enright-Howe-Wallach [3], G is a simple group of hermitian type and one varies a character on the center of K in order to continue the holomorphic discrete series. In [4,5], using the Zuckerman derived functors, these results were extended to cover certain non highest weight modules. Among other results is a proof of unitarity for certain coherent continuations of discrete series representations out of the Borel-de Siebenthal [1] Weyl chamber. These results are tabulated in Appendix 1. They are based on

THEOREM. Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$, say $\mathfrak{q} = \mathfrak{m} + \mathfrak{n}$ where $\mathfrak{n}$ is the nilradical and $\mathfrak{m}$ is a θ-stable Levi component, such that $[\mathfrak{n} \cap \mathfrak{k}, \mathfrak{n}] = 0$ and $\mathfrak{m} = \mathfrak{m}_0 \otimes_{\mathbb{R}} \mathbb{C}$ where $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{g}_0$. Fix a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{m}_0$ of $\mathfrak{g}_0$ and $\lambda \in \mathfrak{h}^*$ such that the irreducible $\mathfrak{m}$-modules $F(\lambda)$ of highest weight λ is finite dimensional and unitarizable. Choose $\zeta \in \mathfrak{h}^*$ corresponding to a central element of $\mathfrak{m}$ and normalized so that (i) the one-dimensional $\mathfrak{m}$-module $F(\zeta)$ is unitarizable, and (ii) $\langle \zeta, \alpha \rangle > 0$ for all roots $\alpha \in \Delta(\mathfrak{n})$. Assume that the relative Verma module

$$N(\lambda + z\zeta) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F(\lambda + z\zeta)$$

is irreducible for $z \leqq c$. Let

$$\Gamma^i : \mathcal{C}(\mathfrak{g}, \mathfrak{m} \cap \mathfrak{k}) \to \mathcal{C}(\mathfrak{g}, \mathfrak{k})$$

denote the i^{th} right derived functor, of the $\mathfrak{k}$-finite submodule functor, from the category of $\mathfrak{g}$-modules that are completely reducible and locally finite for $\mathfrak{m} \cap \mathfrak{k}$. Let $s = \dim \mathfrak{k}/\mathfrak{m} \cap \mathfrak{k}$. Then $\Gamma^s N(\lambda + c\zeta)$ is zero or unitarizable.

See [5] for the proof of this theorem.

This situation arises when $\operatorname{rank} K = \operatorname{rank} G$, the root ordering is such that there is just one noncompact simple root (say α_0), necessarily of coefficient 1 or 2 in the maximal root, and $\mathfrak{q}$ is the maximal parabolic subalgebra of $\mathfrak{g}$ defined by α_0. The hermitian case, coefficient 1, is the case of Enright-Howe-Wallach [3]. Now suppose that α_0 has coefficient 2 in the maximal root and denote

$$\Delta(i) = \{\text{roots } \beta: \ \alpha_0 \text{ has coefficient } i \text{ in } \beta\}$$

so the full root system $\Delta = \Delta(-2) \cup \Delta(-1) \cup \Delta(0) \cup \Delta(1) \cup \Delta(2)$. Then, if

$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ as usual under θ,

$$\mathfrak{m} = \mathfrak{h} + \sum_{\Delta(0)} \mathfrak{g}_\beta \quad , \quad \mathfrak{u} \cap \mathfrak{p} = \sum_{\Delta(1)} \mathfrak{g}_\beta \quad \text{and} \quad \mathfrak{u} \cap \mathfrak{k} = \sum_{\Delta(2)} \mathfrak{g}_\beta$$

$\zeta \in \mathfrak{h}^*$ is normalized by

$$2\langle\zeta,\alpha_0\rangle / \langle\alpha_0,\alpha_0\rangle = 1 \quad \text{and} \quad \langle\zeta,\beta\rangle = 0 \ \text{for all simple} \ \beta \neq \alpha_0 \ .$$

Given λ, we normalize it on the line $\ell = \mathbb{R}\zeta$ by adding a multiple of ζ, so that

$$\lambda + z\zeta + \rho \in C \iff z < 0$$

where C is the $\mathfrak{q}$-antidominant Weyl chamber and ρ as usual is half the sum of the positive roots. The question, now, for applying the theorem is to find the first reduction point, i.e. the smallest number $a = a(\lambda)$ such that $N(\lambda + a\zeta)$ is reducible. Then the theorem says that

> $\Gamma^s N(\lambda + z\zeta)$ is a unitarizable $(\mathfrak{g},K)$-module whenever $z < a$ and $\lambda + z\zeta$ is a $\Delta(\mathfrak{k})$-integral.

That gives quite a number of new irreducible unitary representations, as one sees from the tabulation in Appendix 1, for the case $\dim F(\lambda) = 1$. There, for example, it gives 30 new unitary representations of E_{8,E_7A_1} with singular infinitesimal character, it gives 17 for E_{7,D_6A_1}, and it gives 11 for E_{6,D_5A_1}.

The key to finding the first reduction point is, of course, Jantzen's irreducibility criterion [6] for relative Verma modules. Jantzen's contravariant form — or, equivalently, its hermitian analog — has formal determinant on $N(\lambda')$ given by

$$\prod_{\substack{\text{weights}\\ \text{of } M(\lambda')}} (\text{nonzero}) \prod_{\alpha \in \Delta(\mathfrak{u})} \prod_{\substack{n \text{ positive}\\ \text{integer}}} \left\{\frac{1}{n}\left(\frac{2\langle\lambda' + \rho,\alpha\rangle}{\langle\alpha,\alpha\rangle} - n\right)\right\}^{\chi'(\lambda' - n\alpha)}$$

where $M(\lambda')$ is the ordinary Verma module and where

$$\chi'(\nu) = \sum_{w \in W(\mathfrak{m},\mathfrak{h})} \det(w)\, e^{w(\nu+\rho)} \ .$$

Here $N(\lambda')$ is irreducible if and only if the determinant is nonzero. In other words, if we define

$$\chi(\lambda',\alpha) = \sum_{w \in W(\mathfrak{m},\mathfrak{h})} \det(w)\, e^{w(s_\alpha(\lambda'+\rho))}$$

then $N(\lambda')$ can only reduce when

a) there is at least one root $\alpha \in \Delta(u)$ such that $2\langle \lambda'+\rho,\alpha\rangle / \langle \alpha,\alpha \rangle$ is a positive integer, and

b) the sum over all such roots α, of the $\chi(\lambda',\alpha)$, does not vanish.

Thus we can test $N(\lambda + z\zeta)$ for reduction in a systematic manner, as follows:

1. Compute $2\langle\lambda+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle$ for every $\alpha \in \Delta(\mathfrak{n})$. Note that $2\langle\lambda+z\zeta+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle$ is obtained from it by adding $z \times$ (coefficient of α_0 in α). Note that z must be an integer or half-integer in order for $\lambda+z\zeta+\rho$ to be $\Delta(\mathfrak{k})$-integral, thus in order for $\Gamma^S N(\lambda+z\zeta)$ to be nonzero. Start with z the smallest integer or half-integer such that some $\langle\lambda+z\zeta+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle$, $\alpha \in \Delta(\mathfrak{n})$, is a positive integer.

2. Let $A = \{\alpha\in\Delta(\mathfrak{n})\colon 2\langle\lambda+z\zeta+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle \text{ is a positive integer}\}$. If A is empty then $N(\lambda+z\zeta)$ is irreducible.

3. Suppose A is not empty. Given $\alpha\in A$, suppose there is some $\beta\in\Delta(\mathfrak{n})$ with $\langle\lambda+z\zeta+\rho,\beta\rangle = 0$ and with $s_\alpha(\beta)\in\Delta(\mathfrak{m})$, i.e. with $\langle s_\alpha(\beta),\zeta\rangle = 0$. Then $\langle s_\alpha(\lambda+z\zeta+\rho),s_\alpha(\beta)\rangle = \langle\lambda+z\zeta+\rho,\beta\rangle = 0$ shows $s_\alpha(\lambda+z\zeta+\rho)$ to be $\Delta(\mathfrak{m})$-singular, so $\chi(\lambda+z\zeta,\alpha) = 0$.

4. Let $B = \{\alpha\in A\colon \text{there is no } \beta \text{ as in (3) above}\}$. If B is empty then $N(\lambda+z\zeta)$ is irreducible.

5. Suppose B is not empty. If $\alpha\in B$ then $\chi(\lambda+z\zeta,\alpha)\neq 0$. Thus the only way that we can have $\sum_{\alpha\in\beta}\chi(\lambda+z\zeta,\alpha) = 0$ is if the set B decomposes into pairs $\{\alpha,\alpha'\}$ such that $s_\alpha(\lambda+z\zeta+\rho)$ and $s_{\alpha'}(\lambda+z\zeta+\rho)$ differ by an element $w\in W(\mathfrak{m},\mathfrak{h})$ which has $\det(w) = -1$. Note that, for such a pair, $\langle s_\alpha(\lambda+z\zeta+\rho),\zeta\rangle = \langle ws_\alpha(\lambda+z\zeta+\rho),\zeta\rangle = \langle s_{\alpha'}(\lambda+z\zeta+\rho),\zeta\rangle$, so we need only try to pair off elements $\alpha,\alpha'\in B$ for which $\langle s_\alpha(\lambda+z\zeta+\rho),\zeta\rangle = \langle s_{\alpha'}(\lambda+z\zeta+\rho),\zeta\rangle$.

6. Let $C = \{\alpha\in B\colon \text{there is no } \alpha'\in B \text{ forming a pair } \{\alpha,\alpha'\} \text{ as in (5) above}\}$. Then $N(\lambda+z\zeta)$ is irreducible if and only if C is empty.

7. If $N(\lambda+z\zeta)$ is reducible then $\lambda+z\zeta$ is the first reduction point. If $N(\lambda+z\zeta)$ is irreducible, increase z by $\frac{1}{2}$ and start over again at (2).

In the case $\dim F(\lambda) = 1$, we recently carried out this computational program by hand for the classical simple groups G with rank K = rank G, by arranging the "matrix" $(2\langle\lambda+z\zeta+\rho,\alpha\rangle/\langle\alpha,\alpha\rangle)_{\alpha\in\Delta(\mathfrak{n})}$ in a way that made the various tests relatively straightforward and that made induction arguments convenient. This doesn't work for the exceptional groups, but the structures of types G_2 and F_4 are sufficiently small so that hand calculation was not difficult.

We illustrate the test for reduction by doing the case $G = Sp(1,n-1)$. Let α_i, $1\leq i\leq n$, be the simple roots with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1\leq i<n$, and $\alpha_n = 2\varepsilon_n$. For