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AN EXPLICIT CONSTRUCTION OF THE K - FINITE VECTORS IN THE DISCRETE SERIES FOR AN ISOTROPIC SEMISIMPLE SYMMETRIC SPACE.

BY

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§ 1. Introduction.

In [18] Strichartz stated an explicit formula describing all $O(n)\times O(N)$ - finite functions in any $O(n,N)$ - invariant, closed and irreducible subspace of $L^2(O(n,N)/O(n,N-1))$. From a grouptheoretical point of view the formula is not so transparent since its formulation uses an explicit realization of $O(n,N)/O(n,N-1)$ as a hyperbolic space in $\mathbb{R}^{n+N}$.

In this paper we suggest a formula, which may do the same for the general semisimple symmetric space G/H , i.e. describe the K-finite functions in the irreducible representations of G in $L^2(G/H)$. The socalled discrete series for G/H . For the general case we can only state a few necessary and a few sufficient conditions for our formula to give a K-finite function in a discrete series for G/H . However for the isotropic spaces G/H the formula suffices to describe all the K-types of all the discrete series for G/H .

By the classification, cf. Wolf [22] and Berger [1]. The PseudoRiemannian, nonRiemannian isotropic spaces are all symmetric. Up to coverings they are the classical real-, complex- and quarternionic projective hyperbolic spaces.

$$SO_e(p,q+1)/S(O(p,q)\times O(1))$$
$$SU(p,q+1)/S(U(p,q)\times U(1))$$
$$Sp(p,q+1)/Sp(p,q)\times Sp(1)$$

for $p \geq 1$ and $q \geq 1$, and one exceptional case

$$F_{4(-20)}/Spin(1,8) \ .$$

This last space may in some sense be thought of as the projective hyperbolic space over the Cayley numbers with $p = q = 1$. We are not interested in the Riemannian isotropic spaces, since they have no discrete series.

Our formula, when explicitly computed for a real hyperbolic space, gives a formula very similar to Strichartz',but not completely identical to it. This difference between the two formulas may contain some nontrivial relations between formulas for special functions.

Our interest in the problem came from a discussion of the paper Flensted-Jensen [3]. In that paper it is shown that if rank G/H = rank $K/K\cap H$ then discrete series do exist. The existence is shown by construction of K-finite elements in the corresponding subspaces of $L^2(G/H)$. In Section 8 of [loc. cit.] it was shown by looking at the nonRiemannian isotropic spaces that the construction did not exhaust the discrete series for these spaces, at least for q large compared to p. It is our hope that the present study of these exceptional discrete series for the isotropic spaces may give some hints of how to solve the general problem of construction of all discrete series for G/H.

A recent preprint of Oshima-Matsuki [14] contains very much information on "where to expect" these exceptionel discrete series in general. However the general problem of actual construction of K-finite elements in each representation space seems still open.

We get two offspins of our result: In Section 3 we find a new proof of the minimality of the K-types used in [3] to construct the discrete series for G/H. This proof is simpler than the proof by Schlichtkrull [16] and is also valid for the universal covering space $\tilde{G}/H$, which was not covered by [16], because of the use of results from Vogan-Speh [21]. In Eksample 4.8 the computations show that there are examples where a minimal K-type of a discrete series for G/H does not have a $K \cap H$-fixed vector. In contrast to what for a long time was the belief of the first auther, cf. [4]. Also non-uniqueness of minimal K-types occurs.

The content of the present paper is as follows: In Section 2 we introduce the necessary notation and prerequisits. In Section 3 we introduce and discuss our proposed integral formula. In Section 4 we turn to the case of rank one and in particular to the isotropic spaces. For these our formula gives the complete answer. In Section 5 we compute explicitly the formulas optained in Section 4.

We want to thank Plesner Jakobsen and Schlichtkrull for many fruitfull discussions concerning the content of the present paper.

§ 2. Notation and preliminaries.

Let G be a connected, linear semisimple Lie group contained as a real form in a complex, simply connected Lie Group $G_{\mathbb{C}}$. Let τ be an involution of G and let $H = G_e^\tau$ be the identity component of the fixpoints for τ. Let $X^\sim = G^\sim/H$ be the universal covering space of $X = G/H$ 1). Every connected, simply connected semisimple symmetric space is of the form $X^\sim = G^\sim/H$. For more detailes see Berger [1], Loos [12] or Flensted-Jensen [4].

If $X^\sim$ is irreducible, then X is one of the following three types

(I) The compact type if G is compact.
(II) The noncompact type if H is compact and G is noncompact.
(III) The nonRiemannian type if H in noncompact.

The Killing form induces an invariant metric on X and on $X^\sim$. In case (I) and (II) the metric is Riemannian. In case (III) it is pseudoRiemanninan.

Up to H-conjugacy there is a unique maximal compact subgroup K of G, such that $\tau(K) = K$. Let σ be the Cartan involution related to K. Then $\sigma\tau = \tau\sigma$. Notice that for $X^\sim$ irreducible the three types can be characterized by: (I) $G = K$,. (II) $H = K$ and (III) $G \neq K$ and $H \neq K$.

Examples. (a). A connected semisimple Lie group G_1 may be considered as a symmetric space. Let $d(G_1)$ be the diagonal subgroup in $G_1 \times G_1$, then $G_1 \times G_1/d(G_1)$ is a symmetric space, which as a manifold is isomorphic to G_1. It is of type (I) if G_1 is compact. Otherwise it is of type (III).
(b). The hyperbolic spaces mentioned in Section 1 is of type (I) if $p = 0$, type (II) if $p \geq 1$ and $q = 0$ and type (III) if $p \geq 1$ and $q \geq 1$. □

The Riemannian symmetric spaces are well studied, cf. f.ex. Helgason [6], [8] and [9]. Our main concern in this paper is the nonRiemannian spaces. However as described below we shall make extensive use of a Riemannian symmetric space G^o/H^o "dual" to the nonRiemannian space G/H.

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1) $G^\sim$ is chosen such that the covering map of $G^\sim$ onto G is an isomorphism between the analytic subgroups corresponding to $\mathfrak{h}$, the Lie algebra of H.