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WEIGHTED ORBITAL INTEGRALS ON $SL(2,\mathbb{R})$

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Weighted orbital integrals appear in the adelic version of the Selberg trace formula. They give tempered, but non-invariant, distributions on the local groups. In this paper the general notion of Fourier transform for a non-invariant distribution is discussed. In the case when the local group is $SL(2,\mathbb{R})$ the full Fourier transform of the weighted orbital integral is given. The formula is then interpreted in terms of known properties of weighted orbital integrals.

Les intégrales orbitales à poids figurent dans la formula des traces de Selberg au cas global. Ils donnent des distributions tempérées, mais non invariantes, des groupes locals. L'objet du ce travail est de donner des formules explicites pour la transformée de Fourier des integrales orbitales à poids lorsque le groupe local est $SL(2,\mathbb{R})$. Il faut d'abord préciser la notion de transformée de Fourier d'une distribution non invariante. Enfin on démontre que la formule vérifie les propiétés connues des integrales orbitales à poids.

§1. Introduction.

The adelic version of the Selberg trace formula for rank one groups involves a variety of terms which yield interesting tempered distributions on the various local groups, in particular, on real reductive Lie groups. The calculation of the Fourier transforms of these distributions is an important aspect of the use of the trace formula in the theory of automorphic forms.

There are two main types of distributions which must be studied. The Fourier transforms of the first type, ordinary orbital integrals, were calculated for semisimple Lie groups of real rank one by Sally and Warner [6], and for groups of arbitrary rank by the author [5b]. The second, and less understood, type of distributions are the so-called weighted orbital integrals. For real groups, Arthur computed the Fourier transforms of these weighted orbital integrals restricted to the space of cusp forms [1b,c]. His methods can be generalized to include a larger class of functions in the case that the weighting is not as severe as possible [5a]. Finally, in the real rank one case, Warner has computed the Fourier transform on K-biinvariant functions for a certain limit of weighted orbital integrals [7].

The results presented in this paper give the complete Fourier transform of the weighted orbital integral and its associated singular counterpart for the case of $SL(2,\mathbb{R})$ and represent joint work with J. Arthur and P. Sally. In §2 notation and background information are given and the Fourier inversion formula is stated for regular elements. A sketch of the proof is given, but details will appear elsewhere. In §3 the inversion formula is interpreted in terms of the general properties of weighted orbital integrals proved by Arthur. Also, the Fourier transform of the associated singular distribution is given.

§2. The Fourier Inversion Formula.

Let $G = SL(2,\mathbb{R})$, the group of two-by-two matrices with real entries and determinant one. We will need to consider the following subgroups of G :

$$K = \{ t_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : 0 \leq \theta \leq 2\pi \};$$

$$A = \{ h_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} : t \in \mathbb{R} \};$$

$$A_I = \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \};$$

$$N = \{ n_y = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} : y \in \mathbb{R} \};$$

$$\bar{N} = \{ \bar{n}_y = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} : y \in \mathbb{R} \}.$$

If $x \in G$ is decomposed according to the Iwasawa decomposition as $x = k\bar{n}a$ where $k \in K$, $\bar{n} \in \bar{N}$, and $a \in A$, define $v(x) = v(\bar{n}) = \frac{1}{2}\log(1+y^2)$ if $\bar{n} = \bar{n}_y$ as above. Then v is left K and right A-invariant and is the weighting function used to define the weighted orbital integral which occurs in the Selberg trace formula for $SL(2,\mathbb{R})$. Thus for $f \in C_c^\infty(G)$, the weighted orbital integral of f is the function on the Cartan subgroup $H = A_I A$ defined by

$$(2.1) \quad T_f(\gamma h_t) = |e^t - e^{-t}| \int_{G/A} f(x\gamma h_t x^{-1})v(x)d\dot{x}, \quad \gamma \in A_I,\ t \neq 0 .$$

Here $d\dot{x}$ is a suitably normalized G-invariant measure on the quotient space G/A. It will be useful also to consider the (unweighted) orbital integral or "invariant integral"

$$(2.2) \quad F_f(\gamma h_t) = |e^t - e^{-t}| \int_{G/A} f(x\gamma h_t x^{-1})d\dot{x}, \quad f \in C_c^\infty(G), \quad \gamma \in A_I \quad t \neq 0.$$

The invariant integral was studied by Harish-Chandra and shown to have the following properties [4a,b,c].

(I-1) For fixed $h \in H' = \{\gamma h_t : \gamma \in A_I, t \neq 0\}$, the distribution $F(h): f \longrightarrow \langle F(h),f\rangle = F_f(h)$, $f \in C_c^\infty(G)$, is tempered. That is, it extends continuously to the Schwartz space $\mathcal{C}(G)$.

(I-2) The distributions $F(h)$, $h \in H'$, are invariant. That is, for any $f \in \mathcal{C}(G)$ and $y \in G$, define $f^y \in \mathcal{C}(G)$ by $f^y(x) = f(y\,x\,y^{-1})$, $x \in G$. Then $\langle F(h),f\rangle = \langle F(h),f^y\rangle$.

(I-3) For fixed $f \in \mathcal{C}(G)$, $\gamma \in A_I$, consider the function on $\mathbb{R}\setminus\{0\}$ defined by $t \longmapsto F_f(\gamma:t) = F_f(\gamma h_t)$. It is an even function and tends to zero as $|t| \longrightarrow \infty$. Although initially defined only for $t \neq 0$, it extends to a smooth function on all of $\mathbb{R}$.

(I-4) There is a left and right invariant differential operator z on G so that

$$\frac{d^2}{dt^2} F_f(\gamma h_t) = F_{zf}(\gamma h_t).$$

Arthur has established the following properties of the weighted orbital integral which are similar to those of the invariant integral [1a,b,c,d].

(W-1) For fixed $h \in H'$, the distribution $T(h): f \longrightarrow \langle T(h),f\rangle = T_f(h)$ is tempered.

(W-2) The distributions $T(h)$, $h \in H'$, are not invariant. However, they are K-central. That is, for $k \in K$, $\langle T(h),f\rangle = \langle T(h),f^k\rangle$ for all $f \in \mathcal{C}(G)$. Further, there is a specific non-invariant distribution $T_N(h)$, defined in terms of its Fourier transform, such that $T(h) - T_N(h)$ is invariant.

(W-3) For fixed $f \in \mathcal{C}(G)$, $\gamma \in A_I$, $t \longmapsto T_f(\gamma:t) = T_f(\gamma h_t)$ is an even function, tending to zero at infinity. Although smooth for $t \neq 0$, it is badly behaved at $t = 0$. To describe more exactly its behavior at zero, define $S_f(\gamma:t) = T_f(\gamma:t) + \log(1-e^{-2t})F_f(\gamma:t)$. Then $S_f(\gamma:t)$ is continuous at $t = 0$ and its first derivative has well-defined one-sided limits at zero satisfying $\lim_{t\downarrow 0} \frac{d}{dt} S_f(\gamma:t) - \lim_{t\uparrow 0} \frac{d}{dt} S_f(\gamma:t) = cf(\gamma)$ where c is a constant. The singular weighted orbital integral associated to $T_f(\gamma h_t)$ is defined by