Mémoires de la S. M. F.

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Mémoires de la S. M. F. 2e série, tome 15 (1984), p. 291-305
<http://www.numdam.org/item?id=MSMF_1984_2_15__291_0>

Société Mathématique de France
2e série, Mémoire n° 15, 1984, p. 291-305

THE ASYMPTOTIC BEHAVIOR OF HOLOMORPHIC REPRESENTATIONS

By

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Abstract. In this article the Jacquet module of a holomorphic representation is computed by a direct and elementary method. The preliminary results involve the study of the notion of opposite parabolic subalgebra.

* Research partially supported by an N.S.F. grant.

Introduction.

Let $\underline{g}_o$ be a semi-simple Lie algebra over R. Let $\theta:\underline{g}_o \to \underline{g}_o$ be a Cartan involution of $\underline{g}_o$ and let $\underline{k}_o = \{X \in \underline{g}_o \mid \theta X = X\}$. Let $\underline{g}$ (resp. $\underline{k}$) denote the complexification of $\underline{g}_o$ (resp. $\underline{k}_o$). Then a $(\underline{g}, \underline{k})$ - module, M, is said to be holomorphic if there is a θ-stable Borel subalgebra, $\underline{b}$, such that M is in the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for $\underline{b}$. It is not hard to show that if there exists an infinite dimensional holomorphic $(\underline{g},\underline{k})$-module, then $\underline{g}_o$ contains a θ-stable simple ideal $\underline{g}_o'$ such that $(\underline{g}_o', \underline{k}_o \cap \underline{g}_o')$ is a symmetric pair of Hermitian type.

The purpose of this article is to give a description of the Jacquet module of an irreducible holomorphic representation. No doubt many of the results of this article are known to several specialists in the field (for example, Casselman and Zuckerman have communicated certain less precise results to us). However, there is no place in the literature where one can find a reference. The importance of these results now stems from the fact that the unitarizable holomorphic representations have been classified ([1]).

As it turns out the determination of the Jacquet module of a holomorphic representation is relatively easy once one understands the notion of "opposite parabolic". The first half of this paper is devoted to a rather detailed study of opposite parabolics and the relationship between their categories $\mathcal{O}$. We give here an example for $\underline{s\ell}(2,\mathbb{C}) = \underline{g}$. Let

$$H = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then $\underline{b} = \mathbb{C}H + \mathbb{C}X$ and $\underline{b}' = \mathbb{C}h + \mathbb{C}x$ are opposite parabolics. Indeed

$$\underline{b} \cap \underline{b}' = \mathbb{C}\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix}.$$

Using $\mathbb{C}\begin{pmatrix} 1 & -2i \\ 0 & -1 \end{pmatrix}$ as a Cartan subalgebra of $\underline{g}$ one sees that $\underline{b}'$ is the opposite Borel subalgebra in the usual sense.

HOLOMORPHIC REPRESENTATIONS

As a consequence of our results in the second part of this paper we describe the Langlands parameters of holomorphic representations. Collingwood has informed us that he knows how to do this also (although we have not seen his results or methods).

1. Remarks on the Category $\mathcal{O}$.

Let $\underline{g}$ be a reductive Lie algebra over $\mathbb{C}$. Let $\underline{b} \subset \underline{g}$ be a Borel subalgebra of $\underline{g}$. We define the category $\mathcal{O}'(\underline{b})$ to be the subcategory of the category $\mathcal{M}(\underline{g})$ of $\underline{g}$-modules consisting of those $M \in \mathcal{M}(\underline{g})$ such that

(1) M is finitely generated as a $U(\underline{g})$-module.

(2) If $m \in M$, then $\dim U(\underline{b}) \cdot m < \infty$.

Lemma 1.1. Let $\underline{h} \subset \underline{b}$ be a Cartan subalgebra of $\underline{b}$. If $M \in \mathcal{O}'(\underline{b})$ and $\underline{h}$ acts semi-simply on M then if $\underline{h}' \subset \underline{b}$ is a Cartan algebra of $\underline{g}$ then $\underline{h}'$ acts semi-simply on M.

Proof. Let $\underline{n}(\underline{b})$ be the nil radical of $\underline{b}$. Then there is $X \in \underline{n}(\underline{b})$ such that $e^{\operatorname{ad}X} \cdot \underline{h} = \underline{h}'$. By (2) we see that X acts locally nilpotently on M (i.e. if $m \in M$ there is $k = k(m)$ such that $X^{k(m)} m = 0$). Thus if $t \in \mathbb{C}$ we can form

$$T(t) \cdot m = \sum (t^n/n!) X^n m, \quad m \in M .$$

The sum is actually finite for all $m \in M$. By the obvious formal relations one has

$$T(t+S) = T(t)\, T(S)$$

$$\text{So} \quad T(-1)\, T(1) = T(1)\, T(-1) = I .$$

Set $T = T(1)$. Then T is bijective on M. Also if $Y \in \underline{g}$ then

$$T(t)\, Y m = (e^{t \operatorname{ad}X}\, Y)\, T(t)\, m .$$

Hence

$$Th \cdot m = (e^{\operatorname{ad}X}\, h) Tm, \quad h \in \underline{h} .$$

This clearly implies the result.

We can thus define the category $\mathcal{O}(\underline{b})$ to be the subcategory of $\mathcal{O}'(\underline{b})$ consisting of these objects M of $\mathcal{O}'(\underline{b})$ that are semi-simple relative to some Cartan subalgebra $\underline{h} \subset \underline{b}$.

Let $\mathcal{B}$ denote the space of all Borel subalgebras of $\underline{g}$. If $\underline{b} \in \mathcal{B}$ we denote by $\bar{\mathcal{B}}(\underline{b})$ the subset of all $\bar{\underline{b}} \in \mathcal{B}$ such that $\bar{\underline{b}}$ $\underline{b}$ is a Cartan subalgebra of $\underline{g}$.

We describe $\bar{\mathcal{B}}(\underline{b})$. Let $\underline{h} \subset \underline{b}$ be a Cartan subalgebra. Let $\Phi(\underline{g},\underline{h})$ be the root system of $\underline{g}$ with respect to $\underline{h}$. Let Φ^+ be the system of positive roots of $\Phi(\underline{g},\underline{h})$ corresponding to $\underline{b}$. Then $\underline{n}(\underline{b}) = \bigoplus_{\alpha \in \Phi^+} \underline{g}_\alpha$. Set $\bar{\underline{n}}(\underline{b},\underline{h}) = \sum_{\alpha \in \Phi^+} \underline{g}_{-\alpha}$. Then $\bar{\underline{b}}(\underline{b},\underline{h}) = \underline{h} + \bar{\underline{n}}(\underline{b},\underline{h}) \in \mathcal{B}$. Clearly $\bar{\underline{b}} \cap \underline{b} = \underline{h}$.