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THE ROLE OF RUDIMENTARY RELATIONS IN COMPLEXITY THEORY

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Résumé:

On étudie dans cet article les classes R et XR des relations rudimentaires et faiblement rudimentaires qui se reposent sur la relation de la concaténation bornée . On obtient RUD et XRUD , les classes correspondantes des langages , comme l'union d'une hiérarchie linéaire resp. polynômiale . Ces hiérarchies utilisent des quanteurs alternants aux longueurs bornés ou également des machines alternantes de Turing avec alternance constante . Nous allons introduire une autre description utilisant des quanteurs alternants pour des oracles . En plus on obtiendra une chaîne nouvelle des hiérarchies pour tous les niveaux exponentiels , dont l'union sera ERUD , l'analogue exponentiel de la classe RUD . Et on va montrer que ERUD est la classe E_3 des langages élémentaires .

Abstract:

We shall study the classes R resp. XR of rudimentary resp. extended rudimentary relations which are based on the relation of bounded concatenation . The associated classes RUD resp. XRUD of languages are the union of a linear - resp. polynomial time hierarchy . It can be described either by means of alternating length bounded quantifiers or by means of Turing machines with constant alternation . We shall introduce another description based on alternating quantifiers for oracle sets . Extending these results we obtain a chain of hierarchies for the iterated exponential time levels , whose union is the class ERUD , the exponential analogue of RUD . Moreover , it will be shown that ERUD coincides with the class of elementary recursive languages .

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1. Introduction:

This paper is a survey on the classes R , XR , ER of rudimentary resp. extended rudimentary resp. exponential rudimentary relations and the corresponding classes RUD , XRUD , ERUD of languages . R and XR were introduced by Smullyan in 1961 resp. Bennett in 1962 (cf.[19],[1]) , whereas ER is a new class . As we shall see later , a relation is rudimentary if it is definable from the concatenation relation by means of a first order formula where all quantifiers have linear length bounds . XR resp. ER will be the polynomially resp. exponentially bounded analogue of R .

The associated classes RUD , XRUD , ERUD may be obtained as the union of certain hierarchies . In her thesis in 1975 Wrathall [27] has shown that there are length bounded quantification hierarchies which yield LH = RUD resp. PH = XRUD and have as first step NLTIME resp. NPTIME . As length bounded quantification is closely related to time bounded alternation , these hierarchies can also be described as constant alternation hierarchies for LH and PH (cf.Chandra,Stockmeyer [4],Kozen [10]) .

Recently Orponen [16] has introduced a class EH as the union of an exponential time hierarchy involving oracle set quantification and having NEXPTIME as a first step . Extending his approach we are able to describe the hierarchies for LH and PH as oracle set quantification hierarchies . Moreover , we shall introduce classes $EH^{(i)}$ as the union of an analogous hierarchy involving the i-th iterate e_i of the exponential function , and we shall show that each of the three descriptions may be used . As a consequence we obtain that ERUD is the union of the classes $EH^{(i)}$ and coincides with the class of elementary recursive languages . In addition , the alternating log-space hierarchy of Chandra,Kozen and Stockmeyer [5] may be viewed as step -1 of this chain of hierarchies .

The class $EH^{(i)}$ which consists of languages requiring a constant number of alternations is contained in the class LA_i the corresponding class with a linear amount of alternation . Recently we have shown that the decision problem of the theory e_i-bounded concatenation is complete in the class LA_i w.r.t. polynomial time reductions for $i \geq 1$. In a certain sense these results for $EH^{(i)}$ and LA_i measure the power of e_i-bounded concatenation (cf.also Wilkie [24,25,26]) . However , the question whether the inclusion $EH^{(i)} \subseteq LA_i$ is proper for some $i \geq 0$ remains open . A positive answer would imply that the inclusions PH $\subseteq$ APTIME and LH $\subseteq$ ALTIME are proper , thus solving important open problems in complexity theory .

2. Concatenation as a base of computability theory:

In 1946 Quine [17] suggested to use the concatenation relation rather than addition and multiplication as a base of computability theory . Thus in 1961 Smullyan [19] introduced the class R resp. R_s of rudimentary resp. strictly rudimentary relations on {1,2}* . They consist of those relations which are definable from the concatenation relation by a first order formula where all quantifiers have a linear bound resp. are subword quantifiers. Smullyan has shown that R_s is all we need to describe computations. Each language $L \subseteq \{1,2\}^*$ which is recursively enumerable i.e. accepted by some Turing machine M can be obtained from a relation Q in R_s as follows: $x \in L$ iff $\exists y:(x,y) \in Q$, where $(x,y) \in Q$ expresses the fact that y is an accepting computation sequence with input x. This shows that R_s is large enough to enable us to describe Turing machine computations by means of words consisting of sequences of configuration words. On the other hand R_s is quite small since the associated class RUD_s of languages is contained in LOGSPACE and does not contain $\{1^n 2^n : n \in N\}$ (cf. Nepomnjascii [15], Meloul [11]). In addition, the NPTIME-complete problem SAT(x) is of the form $\exists y: |y| \leq |x| \wedge Q(x,y)$ with Q in R_s as Meloul [11] has shown. This may explain why the class R_s and the related classes R and XR play an important role in complexity theory.

3. The rudimentary relations:

The class R resp. R_s of rudimentary resp. strictly rudimentary relations on $\{1,2\}^*$, introduced by Smullyan [19], is defined as the least class of relations which contains the concatenation relation Con and which is closed under the boolean operations, explicit transformations and linearly bounded resp. subword quantification. The class R^+ of positive rudimentary relations on $\{1,2\}^*$, introduced by Bennett [1], is defined as the least class of relations which contains the relation Con and which is closed under finite unions and intersections, explicit transformations, subword quantification and linearly bounded existential quantification.

$\exists y: y \subseteq x \wedge \ldots$, $\forall y: y \subseteq x \rightarrow \ldots$	subword quantification
$\exists y: \lvert y \rvert \leq k \lvert x \rvert \wedge \ldots$, $\forall y: \lvert y \rvert \leq k \lvert x \rvert \rightarrow \ldots$	linearly bounded quantification

Using the k-adic encoding words over $\{1,\ldots,k\}$ may be identified with natural numbers. Bennett [1] has shown that modulo the dyadic encoding R coincides with the class CA of constructive arithmetic relations on N, which is the analogue of R on N using + and × rather than Con. In addition, CA coincides with the class of bounded arithmetic relations of Harrow [6]. Moreover, the analogues of R resp. R_s resp. R^+ on $\{1,\ldots,k\}^*$ coincide with R resp. R_s resp. R^+ on $\{1,2\}^*$ modulo the k-adic encoding and the dyadic decoding. Using the sequential encoding $\theta(Q)$ of a relation Q one obtains the corresponding classes of languages on $\{1,2,\S\}$: RUD, RUD_s, RUD^+. It can be shown that these classes may be identified with the unary relations in R, R_s, R^+.

Replacing linearly bounded quantification by polynomially bounded quantification (i.e. $\exists y: |y| \leq |x|^k \wedge \ldots$ and $\forall y: |y| \leq |x|^k \rightarrow \ldots$) one obtains the classes of extended rudimentary resp. extended positive rudimentary relations, which were introduced by Bennett [1].

Going a step further we introduce the classes ER resp. ER^+ of exponential rudimentary resp. exponential positive rudimentary relations. They are obtained from

R resp. R^+ by replacing linearly bounded quantification by exponentially bounded quantification (i.e. $\exists y: |y| \leq e_1(|x|^k) \wedge \ldots$ and $\forall y: |y| \leq e_1(|x|^k) \rightarrow \ldots$ with $e_1(n) = 2^n$). Clearly, iterated exponential functions can be used as length bounds as well. - The corresponding classes of languages are denoted by XRUD, $XRUD^+$ resp. ERUD, $ERUD^+$. These classes are related as follows: $RUD_s \subseteq RUD^+ \subseteq RUD$, $XRUD^+ \subseteq XRUD$, $ERUD^+ \subseteq ERUD$ and $RUD^+ \subseteq XRUD^+ \subseteq ERUD^+$, $RUD \subseteq XRUD \subseteq ERUD$.

It should be mentioned that Jones [8] has introduced sublinear analogues of the class R resp. RUD. In particular, he considered a subclass RUD_{log} of LOGSPACE. It is not clear how this class fits into the above set up.

4. Turing machines with constant resp. linear alternation:

Chandra and Stockmeyer [4] and Kozen [10] have extended the concept of nondeterministic Turing machines (NTM's) to alternating Turing machines (ATM's). There is a close connection between alternation and quantification. In particular, hierarchies defined by bounded quantification are closely related to hierarchies defined by constant alternation using the same time bound.

An ATM $\underline{M}$ is a NTM which has 2 disjoint sets of states, the existential and universal states, and a distinguished accepting resp. rejecting state. Configurations and their successor relation are defined as for NTM's. An input w is accepted by $\underline{M}$ (i.e. $w \in L(\underline{M})$), if there exists a finite accepting subtree B of the computation tree of $\underline{M}$ for w. B is accepting, if (1) the root of B is labeled with the input configuration for w, (2) all leaves of B are labeled with accepting configurations, (3) if a node b of B is labeled with an existential (resp. universal) configuration C then at least one (resp. all) successor configurations C' of C must appear as labels of successors b' of b (cf. Berman [2]).

A language L belongs to the alternation class STA(s,t,a), if L is accepted by an ATM $\underline{M}$ such that each w in L posesses an accepting subtree B of depth $\leq t(n)$ and alternation depth $\leq a(n)$ and each configuration in B uses space $\leq s(n)$, where $n=|w|$. We shall use the notation $STA_\exists(s,t,a)$ resp. $STA_\forall(s,t,a)$ to indicate that the input configuration is required to be existential resp. universal. As special cases we obtain the alternating time class $ATIME(t) = STA(-,t,-)$ and the alternating space class $ASPACE(s) = STA(s,-,-)$. The time class with constant alternation CATIME(t) is defined as $\bigcup \langle STA_\exists(-,t,k) : k \in N \rangle$. Similarly the time class with linear alternation LATIME(t) is defined as $STA_\exists(-,t,id)$.

Alternating time bridges the gap between nondeterministic time and deterministic space as Chandra, Kozen and Stockmeyer [5] have shown:

(*) $NTIME(t) \subseteq CATIME(t) \subseteq LATIME(t) \subseteq ATIME(t) \subseteq DSPACE(t)$ for $t \geq id$

(**) ALOGSPACE = PTIME, APTIME = PSPACE, APSPACE = EXPTIME