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*Lagrangian fibrations
on hyperkähler manifolds*

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On a question of Beauville

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

LAGRANGIAN FIBRATIONS ON HYPERKÄHLER MANIFOLDS – ON A QUESTION OF BEAUVILLE

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ABSTRACT. – Let X be a compact hyperkähler manifold containing a complex torus L as a Lagrangian subvariety. Beauville posed the question whether X admits a Lagrangian fibration with fibre L . We show that this is indeed the case if X is not projective. If X is projective we find an almost holomorphic Lagrangian fibration with fibre L under additional assumptions on the pair (X, L) , which can be formulated in topological or deformation-theoretic terms. Moreover, we show that for any such almost holomorphic Lagrangian fibration there exists a smooth good minimal model, i.e., a hyperkähler manifold birational to X on which the fibration is holomorphic.

RÉSUMÉ. – Soit X une variété hyperkählérienne compacte contenant un tore complexe L en tant que sous-variété lagrangienne. A. Beauville a posé la question suivante : la variété X admet-elle une fibration lagrangienne de fibre L ? Nous démontrons que c'est le cas si X n'est pas projective. Si X est projective nous montrons l'existence d'une fibration lagrangienne presque holomorphe de fibre L sous des hypothèses plus restrictives sur la paire (X, L) . Ces hypothèses peuvent se formuler de deux manières : en termes topologiques ou grâce à la théorie des déformations de (X, L) . Par ailleurs, nous démontrons que pour une telle fibration lagrangienne presque holomorphe il y a toujours un bon modèle minimal lisse, c'est-à-dire une variété hyperkählérienne birationnelle à X sur laquelle la fibration est holomorphe.

Introduction

By the classical decomposition theorem of Beauville-Bogomolov, every compact Kähler manifold with vanishing first Chern class admits a finite cover which decomposes as a product of tori, Calabi-Yau manifolds, and hyperkähler manifolds, see e.g., [5, Thm. 1]. While tori are quite well-understood, a classification of Calabi-Yau and hyperkähler manifolds is still far out of reach. Only in dimension 2, where Calabi-Yau and hyperkähler manifolds coincide, the theory of K3-surfaces provides a fairly complete picture.

Let now X be a hyperkähler manifold, that is, a compact, simply-connected Kähler manifold X such that $H^0(X, \Omega_X^2)$ is spanned by a holomorphic symplectic form σ . From a

differential geometric point of view hyperkähler manifolds are Riemannian manifolds with holonomy the full unitary-symplectic group $\mathrm{Sp}(n)$.

An important step in the structural understanding of a manifold is to decide whether there is a fibration $f: X \rightarrow B$ over a complex space of smaller dimension. For hyperkähler manifolds it is known that in case such f exists, it is a *Lagrangian fibration*: $\dim X = 2 \dim B$, and the holomorphic symplectic form σ restricts to zero on the general fibre. Additionally, by the Arnold-Liouville theorem the general fibre is a smooth Lagrangian torus, see Section 1.2 for a detailed discussion.

In accordance with the case of K3-surfaces (and also motivated by mirror symmetry) a simple version of the so-called Hyperkähler SYZ-conjecture⁽¹⁾ asks if every hyperkähler manifold can be deformed to a hyperkähler manifold admitting a Lagrangian fibration. With this as a starting point, an approach to a rough classification of hyperkähler manifolds has been proposed, see e.g., [41]. A more sophisticated version of the SYZ-conjecture is discussed in Section 6.1.

Here we approach the question of existence of a Lagrangian fibration on a given hyperkähler manifold X under a geometric assumption proposed by Beauville [7, Sect. 1.6]:

QUESTION B. – *Let X be a hyperkähler manifold and $L \subset X$ a Lagrangian submanifold biholomorphic to a complex torus. Is L a fibre of a (meromorphic) Lagrangian fibration $f: X \rightarrow B$?*

Building on work of Campana, Oguiso, and Peternell [10] we give a positive answer in case X is not projective.

THEOREM 4.1. – *Let X be a non-projective hyperkähler manifold of dimension $2n$ containing a Lagrangian subtorus L . Then the algebraic dimension of X is n , and there exists an algebraic reduction $f: X \rightarrow B$ of X that is a holomorphic Lagrangian fibration with fibre L .*

In the case of *projective* hyperkähler manifold X containing a Lagrangian subtorus L , we work out a necessary and sufficient criterion for the existence of an almost holomorphic fibration with fibre L , i.e., for a slightly weaker positive answer to Beauville's question.

THEOREM 5.3. – *Let X be a projective hyperkähler manifold and $L \subset X$ a Lagrangian subtorus. Then the following are equivalent.*

1. X admits an almost holomorphic Lagrangian fibration with strong fibre L .
2. The pair (X, L) admits a small deformation (X', L') with non-projective X' .
3. There exists an effective divisor D on X such that $c_1(\mathcal{O}_X(D)|_L) = 0 \in H^{1,1}(L, \mathbb{R})$.

Here, *strong fibre* means that f is holomorphic near L , and L is a fibre of the corresponding holomorphic map. The proof of Theorem 5.3 consists of two major steps: First, assuming the existence of a small deformation of (X, L) to a non-projective pair (X', L') , we use Theorem 4.1 to produce a Lagrangian fibration with fibre L' on X' and then degenerate this fibration to an almost-holomorphic fibration on (X, L) using relative Barlet spaces. Second, the existence of a small deformation to a non-projective pair (X', L') is characterised in terms

⁽¹⁾ We refer the reader to [42] for a historical discussion concerning the emergence of this conjecture.

of periods in $H^2(X, \mathbb{C})$. This finally leads to the condition on the existence of a special divisor, as stated in part (iii) of Theorem 5.3.⁽²⁾

From the discussion above the question arises how far an almost holomorphic fibration is away from answering Beauville’s question in the strong form. If $f: X \dashrightarrow B$ is an almost holomorphic Lagrangian fibration, then it is natural to search for a holomorphic model of f in the same birational equivalence class. This is done in the final section, where using the recent advances in higher-dimensional birational geometry ([8, 19]) the following result is proven.

THEOREM (see Theorem 6.3). – *Let X be a projective hyperkähler manifold with an almost holomorphic Lagrangian fibration $f: X \dashrightarrow B$. Then there exists a holomorphic model for f on a birational hyperkähler manifold X' . In other words, there is a commutative diagram*

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ f \downarrow & & \downarrow f' \\ B & \dashrightarrow & B' \end{array}$$

where f' is a holomorphic Lagrangian fibration on X' and the horizontal maps are birational.

Theorem 6.3 proves a special version of the Hyperkähler SYZ-conjecture. Related results were obtained by Amerik and Campana [1, Thm. 3.6] in dimension four. Note furthermore that birational hyperkähler manifolds are deformation-equivalent by work of Huybrechts [20, Thm. 4.6], so Theorem 6.3 might also lead to a new approach to the general case of the Hyperkähler SYZ-conjecture.

The connection to this circle of ideas is also manifest in the following generalization of a result of Matsushita, which we obtain as a corollary of Theorem 6.3.

THEOREM 6.12. – *Let X be a projective hyperkähler manifold and $f: X \dashrightarrow B$ an almost holomorphic map with connected fibres onto a normal projective variety B . If $0 < \dim B < \dim X$, then $\dim B = \frac{1}{2} \dim X$, and f is an almost holomorphic Lagrangian fibration.*

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⁽²⁾ After this article was written, Jun-Muk Hwang and Richard Weiss posted a preprint [23] in which they prove that the criterion given in part (iii) of Theorem 5.3 is fulfilled for any projective hyperkähler manifold X containing a Lagrangian subtorus L . See also Remark 6.14.

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1. Preliminaries on hyperkähler manifolds

We collect a few basic definitions and properties of the objects of our study.

DEFINITION 1.1. – An *irreducible holomorphic symplectic manifold* or *hyperkähler manifold* is a simply-connected compact Kähler manifold X such that $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate holomorphic two-form σ .

Actually, the notion of hyperkähler manifold is of differential-geometric origin and stands for a Ricci-flat Kähler manifold with holonomy group $\mathrm{Sp}(n)$. It was shown by Beauville in [5, Prop 4] that this condition is equivalent to the existence of a holomorphic symplectic form unique up to scalars; often the terms *irreducible holomorphic symplectic manifold* and *hyperkähler manifold* are therefore used synonymously.

1.1. The Beauville-Bogomolov form

The second cohomology $H^2(X, \mathbb{Z})$ of a hyperkähler manifold X carries a natural, integral symmetric bilinear form

$$q = q_X : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z},$$

the so-called *Beauville-Bogomolov-Fujiki form* (see [5, Thm. 5] or [21, Def. 22.8]). Since we need to consider the restriction of this form to subspaces where it might be degenerate, we give its signature as a triple containing (in this order) the number of positive, zero, and negative eigenvalues of the associated real symmetric bilinear form. In this notation q has signature $(3, 0, b_2(X) - 3)$, and its restriction to $H^{1,1}(X, \mathbb{R})$ has signature $(1, 0, h^{1,1} - 1)$, see [21, Cor. 23.11].

Let $\rho = \rho(X)$ be the Picard number of X , that is, the rank of the Néron-Severi group $\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Q})$. We distinguish hyperkähler manifolds according to the signature of the restriction of q to $\mathrm{NS}(X)$. We call X *hyperbolic* if $q|_{\mathrm{NS}(X)}$ has signature $(1, 0, \rho - 1)$, *parabolic* if $q|_{\mathrm{NS}(X)}$ has signature $(0, 1, \rho - 1)$, and *elliptic* if $q|_{\mathrm{NS}(X)}$ has signature $(0, 0, \rho)$. The relevance of these notions is underlined by the following result of Huybrechts.

THEOREM 1.2 (Prop. 26.13 of [21]). – *A hyperkähler manifold X is projective if and only if X is hyperbolic.*