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VON NEUMANN AND BIRKHOFF ERGODIC THEOREMS FOR NEGATIVELY CURVED GROUPS

BY Lewis BOWEN* AND Amos NEVO†

ABSTRACT. – We prove maximal inequalities for concentric ball and spherical shell averages on a general Gromov hyperbolic group, in arbitrary probability preserving actions of the group. Under an additional condition, satisfied for example by all groups acting isometrically and properly discontinuously on CAT(-1) spaces, we prove a pointwise ergodic theorem with respect to a sequence of probability measures supported on concentric spherical shells.

RÉSUMÉ. – Pour tout groupe hyperbolique au sens de Gromov et pour toute action, préservant la mesure, sur un espace de probabilités, nous démontrons une inégalité maximale pour les moyennes sur des boules concentriques ou sur des anneaux sphériques concentriques de même épaisseur. Sous une hypothèse supplémentaire, valable par exemple pour les actions isométriques et proprement discontinues sur des espaces CAT(-1), nous démontrons de plus un théorème ergodique ponctuel pour une suite de mesures de probabilités à support dans des anneaux sphériques concentriques.

1. Introduction

1.1. Motivation and background

1.1.1. *The Arnol'd-Krylov problem.* – Given a dynamical system with invariant probability measure, von-Neumann's mean ergodic theorem [40] and Birkhoff's pointwise ergodic theorem [8] assert that the time evolution of the dynamical system distributes it evenly in the space. More concretely, under the sole assumption of ergodicity, namely the absence of invariant sets, when one samples the values of a function on the space at regular time intervals along the trajectory of the dynamical system, the average of these samples over time will converge (in the mean and pointwise) to the space average of the function, in accordance with Bolzmann's "ergodic hypothesis". Equivalently, the trajectories spend the right fraction of time in every subset of space, given by the measure of the subset.

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The study of dynamical systems defined by a measure-preserving transformation is based on two main ingredients, namely that intervals are asymptotically invariant under translations, and that they satisfy the doubling property. Asymptotic invariance arguments play a crucial role in several proofs of Birkhoff's pointwise ergodic theorem, in Riesz's proof of von-Neumann's mean ergodic theorem, in the Krylov-Boglyobov proof of the existence of probability measures invariant under a continuous transformation of a compact space, in Calderón's transference principle reducing aspects of analysis of the orbits of the flow on phase space to analysis on the integers, and in Furstenberg's correspondence principle. The doubling property plays a crucial role in Wiener's covering argument which implies the Hardy-Littlewood maximal inequality, and thus also in the maximal ergodic theorem and the pointwise ergodic theorem. These arguments were extended over the years to actions by any finite number of commuting measure-preserving transformations.

Half a century ago, Arnol′d and Krylov [7] have raised the problem of establishing the following generalization of the classical ergodic theorems. Let Γ be a finitely generated group, and S a finite symmetric generating set. Consider the associated left invariant word metric on Γ, and let B_n and S_n denote the ball and the sphere of radius n with center e. Denote the uniform average on the ball by β_n, and on the sphere by σ_n. Given a measure-preserving action of Γ, consider sampling the values of a function on an orbit of Γ according to β_n. Does this averaging process converge in the mean and pointwise, and if so, what is its limit? Furthermore, when does the averaging process associated with σ_n converge ? Thus the Arnol′d-Krylov problem amounts to establishing ergodic theorems for *any choice* of finitely many measure-preserving transformations.

The Arnol′d-Krylov problem has proved to be very difficult and remains wide open. The only class of groups where a complete positive solution for β_n has been obtained is that of groups with polynomial volume growth. The sequence of balls in such groups satisfies the doubling condition, and the main ingredient in the relatively recent proof of the pointwise ergodic theorem is the fact that the sequence of word metric balls B_n is in fact asymptotically invariant under translations. This was established by Tessera [47] (see also [26]) and in a sharper form by Breuillard [18]. We refer to [42] and [3] for a detailed account of the proof of the pointwise ergodic theorem in this case.

The remarkable utility of asymptotically invariant sequences (exemplified very briefly above) resulted in the fact that much of the effort in developing ergodic theorems for countable groups has been devoted to generalizing the classical convergence results for a single transformation to ergodic actions of groups possessing an asymptotically invariant sequence. This class coincides with the class of amenable groups, and the averages studied most extensively have been uniform averages on a Følner sequence, often satisfying suitable regularity conditions introduced by Tempelman [46], generalizing the doubling condition. For more on the subject of ergodic theorems on amenable groups we refer to the foundational paper by Ornstein and Weiss [44], and for more recent results to Lindenstrauss's ergodic theorem for tempered Følner sequences [39] as well as the survey [49].

However, already for meta-Abelian (non-nilpotent) solvable groups, and certainly for non-amenable groups, the sequence of word-metric balls is not asymptotically invariant, so the methods developed for Følner sequences have not led to further progress on the Arnol′d-Krylov problem.

1.1.2. *Free groups*. – Turning to the problem of establishing ergodic theorems for non-amenable groups, we note that an important special case that figures prominently in the theory is that of free (non-Abelian) groups. In the case of the word metric associated with free generators, the symmetry inherent in the radial structure implies that the convolution algebra generated by σ_n is commutative, and this fact opens the door to a spectral approach to the problem. This approach was initiated already by Arnol′d and Krylov [7] who proved an equidistribution theorem for radial averages on dense free subgroups of isometries of the unit sphere $\mathbb{S}^2$ via a spectral argument similar to Weyl's equidistribution theorem on the circle. Guivarc'h has established a mean ergodic theorem for radial averages on the free group, using von-Neumann's original approach via the spectral theorem [36]. The pointwise ergodic theorem for the averages $\frac{1}{2}(\sigma_n + \sigma_{n+1})$ in general actions of the free group was proved in [41] for L^2-functions using spectral theory, and extended to function in L^p, $p > 1$, in [43], using more refined spectral methods.

Another successful approach to ergodic theorems on free groups and more general Markov groups is based on the theory of Markov operator. Grigorchuk [34] has applied the Hopf-Dunford-Scwartz operator ergodic theorem to deduce a pointwise ergodic theorem for the uniform averages $\mu_k = \frac{1}{k+1}\sum_{n=0}^{k} \sigma_n$ of the spheres. This approach was subsequently generalized by Bufetov to weighted averages of spheres on general Markov groups [20]. Using Rota's approach to the operator ergodic theorem via martingale theory, Bufetov [21] has extended pointwise convergence of the averages $\frac{1}{2}(\sigma_n + \sigma_{n+1})$ on the free group to the function space $L \log L$. Pointwise almost sure convergence for the uniform averages μ_k of the spheres on general Markov groups for bounded functions has been established recently in [22, Cor. 1], and under additional assumption also in [45]. It is a reflection of the difficulty of the Arnol′d-Krylov problem that in both of these results, the limit has not been identified, in general. Under a suitable mixing assumption on the action, it was shown in [32] that the limit is indeed the space average. Finally, we note that an ergodic theorem for actions of general word-hyperbolic groups on finite spaces was established in [12].

1.1.3. *Ergodic theorems for lattice subgroups*. – An important extension of the Arnol′d-Krylov problem is to consider balls and spheres defined by natural left-invariant metrics on the group, not necessarily given by word-metrics. Thus, when the group Γ is a discrete subgroup of a locally compact group, one can consider left-invariant metrics on G restricted to Γ. Extending the scope of the problem gives rise to many natural examples, of which we mention the following. When Γ is a lattice subgroup of a (non-compact) simple Lie group with finite center, consider (any) non-trivial linear representation $\tau : G \to SL_n(\mathbb{R})$, and restrict (any) norm on $SL_n(\mathbb{R})$ to $\tau(\Gamma)$. It was shown in [33] that in any ergodic action of Γ, the associated ball averages converge in norm and pointwise for any function $f \in L^p$, provided $1 < p < \infty$. Similar results hold more generally for irreducible lattices in S-algebraic groups, and we refer to [33] for further details. We remark that the methods of [33] rely in a crucial manner on spectral theory, namely on the unitary representation theory of the semisimple group G involved, and are thus limited only to those countable groups which arise as lattice subgroups of G.

More generally, when Γ acts isometrically and properly discontinuously on a locally compact metric space (X, d), one can consider the (pseudo) metrics given by