Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

WHEN IS A RIESZ DISTRIBUTION A COMPLEX MEASURE?

Alan D. Sokal

Tome 139
Fascicule 4

2011

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre national de la recherche scientifique

pages 519-534

Bull. Soc. math. France
139 (4), 2011, p. 519–534

WHEN IS A RIESZ DISTRIBUTION A COMPLEX MEASURE?

by Alan D. Sokal

Abstract. — Let $\mathscr{R}_\alpha$ be the Riesz distribution on a simple Euclidean Jordan algebra, parametrized by $\alpha \in \mathbb{C}$. I give an elementary proof of the necessary and sufficient condition for $\mathscr{R}_\alpha$ to be a locally finite complex measure (= complex Radon measure).

Résumé (*Une distribution de Riesz, quand est-elle mesure complexe ?*)
Soit $\mathscr{R}_\alpha$ la distribution de Riesz sur une algèbre de Jordan euclidienne simple, paramétrisée par $\alpha \in \mathbb{C}$. Je donne une démonstration élémentaire de la condition nécessaire et suffisante pour que $\mathscr{R}_\alpha$ soit une mesure complexe localement finie (= mesure de Radon complexe).

1. Introduction

In the theory of harmonic analysis on Euclidean Jordan algebras (or equivalently on symmetric cones) [12], a central role is played by the *Riesz distributions* $\mathscr{R}_\alpha$, which are tempered distributions that depend analytically on a

Texte reçu le 16 juin 2009, révisé et accepté le 20 mai 2010.

Alan D. Sokal, Department of Physics, New York University, 4 Washington Place, New York, NY 10003 USA and Department of Mathematics, University College London, London WC1E 6BT, England • *E-mail :* `sokal@nyu.edu`

2000 Mathematics Subject Classification. — 43A85; 17A15, 17C99, 28C10, 44A10, 46F10, 47G10, 60E05, 62H05.

Key words and phrases. — Riesz distribution, Jordan algebra, symmetric cone, Gindikin's theorem, Wallach set, tempered distribution, positive measure, Radon measure, relatively invariant measure, Laplace transform.

parameter $\alpha \in \mathbb{C}$. One important fact about the Riesz distributions is the necessary and sufficient condition for positivity, due to Gindikin [13]:

THEOREM 1.1. — [12, Theorem VII.3.1] *Let V be a simple Euclidean Jordan algebra of dimension n and rank r, with $n = r + \frac{d}{2}r(r-1)$. Then the Riesz distribution $\mathcal{R}_\alpha$ on V is a positive measure if and only if $\alpha = 0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}$ or $\alpha > (r-1)\frac{d}{2}$.*

The "if" part is fairly easy, but the "only if" part is reputed to be deep [13, 12, 20].(1)

The purpose of this note is to give a completely elementary proof of the "only if" part of Theorem 1.1, and indeed of the following strengthening:

THEOREM 1.2. — *Let V be a simple Euclidean Jordan algebra of dimension n and rank r, with $n = r + \frac{d}{2}r(r-1)$. Then the Riesz distribution $\mathcal{R}_\alpha$ on V is a locally finite complex measure [= complex Radon measure] if and only if $\alpha = 0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}$ or $\operatorname{Re}\alpha > (r-1)\frac{d}{2}$.*

This latter result is also essentially known [18, Lemma 3.3], but the proof given there requires some nontrivial group theory.

The idea of the proof of Theorem 1.2 is very simple: A distribution defined on an open subset $\Omega \subset \mathbb{R}^n$ by a function $f \in L^1_{\mathrm{loc}}(\Omega)$ can be extended to all of $\mathbb{R}^n$ as a locally finite complex measure only if the function f is locally integrable also at the boundary of Ω (Lemma 2.1); furthermore, this fact survives analytic continuation in a parameter (Proposition 2.3). In the case of the Riesz distribution $\mathcal{R}_\alpha$, a simple computation using its Laplace transform (Lemma 3.4) plus a bit of extra work (Lemma 3.5) allows us to determine the allowed set of α, thereby proving Theorem 1.2.

Theorem 1.2 thus states a necessary and sufficient condition for $\mathcal{R}_\alpha$ to be a distribution of order 0. It would be interesting, more generally, to determine the order of the Riesz distribution $\mathcal{R}_\alpha$ for each $\alpha \in \mathbb{C}$.

It would also be interesting to know whether this approach is powerful enough to handle the multiparameter Riesz distributions $\mathcal{R}_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r$ [12, Theorem VII.3.2] and/or the Riesz distributions on homogeneous cones that are not symmetric (i.e. not self-dual) and hence do not arise from a Euclidean Jordan algebra [13, 20].

In an Appendix I comment on a beautiful but little-known elementary proof of Theorem 1.1 — which does not extend, however, to Theorem 1.2 — due to Shanbhag [27] and Casalis and Letac [9].

(1) The set of values of α described in Theorem 1.1 is the so-called *Wallach set* [29, 30, 21, 10, 11, 12].

2. A general theorem on distributions

We assume a basic familiarity with the theory of distributions [26, 19] and recall some key notations and facts.

For each open set $\Omega \subseteq \mathbb{R}^n$, we define the space $\mathcal{D}(\Omega)$ of C^∞ functions having compact support in Ω, the corresponding space $\mathcal{D}'(\Omega)$ of distributions, and the space $\mathcal{D}'^k(\Omega)$ of distributions of order $\le k$. In particular, the space $\mathcal{D}'^0(\Omega)$ consists of the distributions that are given locally (i.e. on every compact subset of Ω) by a finite complex measure.

Let $f\colon \Omega \to \mathbb{C}$ be a measurable function, and extend it to all of $\mathbb{R}^n$ by setting $f \equiv 0$ outside Ω. We say that $f \in L^1_{\mathrm{loc}}(\Omega)$ if, for every $x \in \Omega$, f is (absolutely) integrable on some neighborhood of x. Any $f \in L^1_{\mathrm{loc}}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'^0(\Omega)$ by

$$T_f(\varphi) = \int \varphi(x)\, f(x)\, dx \qquad \text{for all } \varphi \in \mathcal{D}(\Omega)\,. \tag{1}$$

We are interested in knowing under what circumstances the distribution $T_f \in \mathcal{D}'^0(\Omega)$ can be extended to a distribution $\widetilde{T}_f \in \mathcal{D}'^0(\mathbb{R}^n)$, i.e. one that is locally everywhere on $\mathbb{R}^n$ a finite complex measure.

Lemma 2.1. — *Let $f\colon \Omega \to \mathbb{C}$ be in $L^1_{\mathrm{loc}}(\Omega)$, and let $T_f \in \mathcal{D}'^0(\Omega)$ be the corresponding distribution. Then the following are equivalent:*

(a) *$f \in L^1_{\mathrm{loc}}(\overline{\Omega})$, i.e. for every $x \in \overline{\Omega}$, f is integrable on some neighborhood of x.*[2]

(b) *There exists a distribution $\widetilde{T}_f \in \mathcal{D}'^0(\mathbb{R}^n)$ that extends T_f and is supported on $\overline{\Omega}$.*

(c) *There exists a distribution $\widetilde{T}_f \in \mathcal{D}'^0(\mathbb{R}^n)$ that extends T_f.*

Proof. — (a) $\Longrightarrow$ (b): It suffices to define $\widetilde{T}_f(\varphi) = \int_\Omega \varphi(x)\, f(x)\, dx$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

(b) $\Longrightarrow$ (c) is trivial.

(c) $\Longrightarrow$ (a): By hypothesis, for every $x \in \partial\Omega$ and every compact neighborhood $K \ni x$, there exists a finite complex measure μ_K supported on K such that $\widetilde{T}_f(\varphi) = \int \varphi\, d\mu_K$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with support in K. But since $\widetilde{T}_f$ extends T_f, the restriction of μ_K to every compact subset of $K \cap \Omega$ must coincide with the measure $f(x)\, dx$. Since $K \cap \Omega$ is σ-compact, this implies that $\int\limits_{K\cap\Omega} |f(x)|\, dx = |\mu_K|(K \cap \Omega) < \infty$, so that f is integrable in a neighborhood of x. □

(2) Since this has already been assumed for $x \in \Omega$, the content of hypothesis (a) is that it should hold also for $x \in \partial\Omega$.

We now extend this idea to allow for analytic dependence on a parameter. Let Ω be an open set in $\mathbb{R}^n$, let D be a connected open set in $\mathbb{C}^m$, and let $F\colon \Omega \times D \to \mathbb{C}$ be a continuous function such that $F(x,\,\cdot\,)$ is analytic on D for each $x \in \Omega$. Then, for each $\lambda \in D$, define

$$T_\lambda(\varphi) \;=\; \int \varphi(x)\, F(x,\lambda)\, dx \qquad \text{for all } \varphi \in \mathscr{D}(\Omega)\ . \tag{2}$$

LEMMA 2.2. — *With F as above, the map $\lambda \mapsto T_\lambda$ is analytic from D into $\mathscr{D}'(\Omega)$ in the sense that $\lambda \mapsto T_\lambda(\varphi)$ is analytic for all $\varphi \in \mathscr{D}(\Omega)$.*

Proof. — This is an immediate consequence of the hypotheses on F together with standard facts about scalar-valued analytic functions in $\mathbb{C}$ (either Morera's theorem or the Cauchy integral formula) and $\mathbb{C}^m$ (e.g. the weak form of Hartogs' theorem). ☐

Remark. — Weak analyticity in the sense used here is actually *equivalent* to strong analyticity: see e.g. [15, pp. 37–39, Théorème 1 and Remarque 1] [5, Theorems 3.1 and 3.2] [14, Theorem 1]. Indeed, our hypothesis on F is equivalent to the even stronger statement that the map $\lambda \mapsto F(\,\cdot\,,\lambda)$ is analytic from D into the space $C^0(\Omega)$ of continuous functions on Ω, equipped with the topology of uniform convergence on compact subsets [15, p. 41, example (a)]. But we do not need any of these facts; weak analyticity is enough for our purposes.

Putting together these two lemmas, we obtain:

PROPOSITION 2.3. — *Let F be as above, let $D_0 \subseteq D$ be a nonempty open set, and let $\lambda \mapsto \widetilde{T}_\lambda$ be a (weakly) analytic map of D into $\mathscr{D}'(\mathbb{R}^n)$ such that $\widetilde{T}_\lambda$ extends T_λ for each $\lambda \in D_0$. Then, for each $\lambda \in D$, we have:*

(a) *$\widetilde{T}_\lambda$ extends T_λ.*
(b) *If $\widetilde{T}_\lambda \in \mathscr{D}'^0(\mathbb{R}^n)$, then $F(\,\cdot\,,\lambda) \in L^1_{\mathrm{loc}}(\overline{\Omega})$.*

Proof. — (a) This is immediate by analytic continuation: for each $\varphi \in \mathscr{D}(\Omega)$, both $\widetilde{T}_\lambda(\varphi)$ and $T_\lambda(\varphi)$ are (by hypothesis and Lemma 2.2, respectively) analytic functions of λ on D that coincide on D_0, therefore they must coincide on all of D.

(b) This is immediate from (a) together with Lemma 2.1. ☐

We shall apply this setup with $F(x,\lambda) = f(x)^\lambda$ where $f\colon \Omega \to (0,\infty)$ is a continuous function; in fact, we shall take f to be a polynomial.