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# ZERO CYCLES AND THE NUMBER OF GENERATORS OF AN IDEAL

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(to Pierre Samuel)

## RESUME

Soit $X$ une surface dans l'espace affine $\mathbb{A}^4$ sur un corps algébriquement clos $k$. On montre que $X$ est ensemblistement intersection complète si $k = \overline{\mathbb{F}}_p$ ou si $X$ n'est pas birationnelle à une surface projective de type général.

On donne aussi des exemples de variétés affines lisses de dimension $n$ qui ne sont pas des sous–variétés fermées dans $\mathbb{A}^{2n}$. La plupart des résultats s'appuie sur les théorèmes de Mumford et de Roitman concernant le groupe de Chow $CH_0(X)$.

## ABSTRACT

Let $X$ be a local complete intersection surface in $\mathbb{A}^4$ over an algebraically closed field $k$. We show that $X$ is set–theoretic complete intersection if $k = \overline{\mathbb{F}}_p$ or if $X$ is smooth and not birational to a surface of general type.

We also give examples of smooth affine varieties of dimension $n$, not admitting a closed immersion in $\mathbb{A}^{2n}$. Most of the results here depend crucially on the results of Mumford and Roitman on the Chow group $CH_0(X)$.

**Introduction.**

Let $k$ be a field and $X$ a closed codimension two local complete intersection sub–scheme of the affine $n$–space $\mathbb{A}^n_k$. Let $I$ be the defining ideal of $X$ in $k[X_1,\dots,X_n]$ and suppose that there is a surjection $I/I^2 \twoheadrightarrow \omega_X$, where $\omega_X$ is the dualizing module of $X$. Then, the Ferrand–Szpiro Theorem ([Sz], see Cor. 0.2 below) asserts that $X$ is a set–theoretic complete intersection. When $X$ is a curve of dimension one, the surjection $I/I^2 \twoheadrightarrow \omega_X$ always exists and thus Ferrand–Szpiro showed that a local complete intersection curve in $\mathbb{A}^3_k$ is a set–theoretic complete intersection. The question whether any local complete intersection sub–scheme of $\mathbb{A}^n_k$ is a set–theoretic complete intersection is open.

In sections 1 and 2, we examine this question for surfaces in $\mathbb{A}^4$. It is shown that local complete intersection surfaces in $\mathbb{A}^4_{\mathbb{F}_p}$ are set–theoretic complete intersections.

For a smooth surface $X$ in $\mathbb{A}^4_k$ ($k$ algebraically closed), the existence of a surjection $I/I^2 \twoheadrightarrow \omega_X$ turns out to be equivalent to the vanishing of $c_1^2$ ($c_1 = c_1(\Omega^1_X)$) in the Chow group of zero–cycles. In view of this, it follows by looking at the classification of surfaces, that if $X$ is not birationally equivalent to a surface of general type, then $X$ is a set–theoretic complete intersection (Th. 2.9). We also show that for a smooth affine variety $X$ in $\mathbb{A}^n_k$, the ideal $I_X$ of $X$ in $k[X_1,\dots,X_n]$ is generated by $n-1$ element if and only if $\Omega^1_X$ has a free direct summand of rank one (Th. 1.11).

In section 3, we give a partial converse to the Ferrand–Szpiro theorem. More precisely, we show that if $X \subset \mathbb{A}^4$ is a smooth surface which is an intersection of two surfaces $F_1 = F_2 = 0$ such that at each point of $X$ either $F_1$ or $F_2$ is smooth, then $c_1^2 = 0$ (Cor. 3.7). In section 4, we prove a result about zero–cycles on the product of two curves, which enables us to produce examples of surfaces $X = C_1 \times C_2$, with $C_i$ smooth affine curves such that $X$ does not admit a closed immersion in $\mathbb{A}^4$. Further for this example $\Omega^1_X$ is not generated by three elements and hence $X$ cannot be immersed in $\mathbb{A}^3$. In section 3, for all $n,d$ with $1 \le d \le n \le 2d+1$ we make examples of smooth $d$–dimensional affine varieties $X$ such that $X$ admits a closed immersion in $\mathbb{A}^n$, but not in $\mathbb{A}^{n-1}$. Further for any embedding of $X$ in $\mathbb{A}^m$, the prime ideal $I(X)$ of $X$ is not generated by $m-1$ element. When $d=2$ this also provides an example of a smooth surface in $\mathbb{A}^4$ with $c_1^2 \neq 0$. The example in sections 4 and 5 are constructed by showing that the appropriate obstructions in zero–dimensional Chow groups do not vanish.

In this paper we use extensively the results of Roitman ([Ro 1], [Ro 2], [Ro 3]) and Mumford ([Mum]) on the Chow group of zero–cycles. In section 5, we need a result about embedding of affine varieties (Th. 5.7). The simple and elegant proof of this theorem we have included here is due to M.V. Nori. Our thanks are due to him for this proof which replaces our earlier lengthly proof of Theorem 5.7. Thanks are also due to V. Srinivas for asking us a question about embedding of affine varieties. Results in section 5 were rewritten and refined recently in response to his question.

The work in this paper began in 1977. A part of this work was outlined in the survey article [Mu 3]. A major portion of this work was done in 1978 when the first and second named authors were visiting IHES and Ecole Normale Supérieure at Paris, respectively, and the third named author was at Ecole Normale Supérieure. We are grateful to these institutions for hospitality and support. The first two authors were also supported by NSF grants.

We have mentioned some of the recent work relevant to this paper in the form of "remarks".

## §0. Notations and preliminaries.

We consider only commutative noetherian rings. Let $A$ be such a ring and $I \subset A$ an ideal. We recall that $I$ is a *complete intersection of height* $r$ *if* $I$ is generated by an $A$–regular sequence of length $r$. The ideal $I$ is a *local complete intersection of height* $r$ if for all maximal ideals $M$ containing $I$, the ideal $I_M \subset A_M$ is a complete intersection of height $r$. The ideal $I$ is a *set-theoretic complete intersection of height* $r$ if there is an ideal $J$ such that $\sqrt{J} = \sqrt{I}$ and $J$ is a complete intersection of height $r$. If $I \subset A$ is a local complete intersection of height $r$, we write $\omega_1 = \mathrm{Ext}^r_A(A/I,A)$. It is well known that $\omega_1 \simeq \mathrm{Hom}(\Lambda^r I/I^2, A/I)$. Note that if $X$ is a smooth affine variety and $V \subset X$ is a local complete intersection sub–scheme of codimension $r$ and $I$ the defining ideal of $V$ in the coordinate ring $A$ of $V$, then $\omega_I$ is the module of sections of $\omega_V \otimes \omega_X^{-1}$, where $\omega_V$ and $\omega_X$ are the canonical sheaves of $V$ and $X$ respectively.

We recall the following result of Ferrand–Szpiro [Sz], which is crucial for this paper.

**Theorem 0** (*Ferrand-Szpiro*). *Let $A$ be a commutative noetherian ring and $I \subset A$ local complete intersection ideal of height* 2. *Suppose there is a surjection $I \twoheadrightarrow \omega_I$. Then there is an exact sequence $0 \to A \to P \to J \to 0$, with $P$ a projective $A$-module of rank* 2 (*for proof see* [Sz] *or* [Mu 2]).

For a projective $R$–module $L$ of rank 1, we write $L^n = L^{\otimes n}$, $L^{-n} = \mathrm{Hom}(L^n,R)$, $n \geq 0$.

**Remark** 0.1. The existence of surjection $I \twoheadrightarrow \omega_I$ is easily seen to be equivalent to the isomorphism $I/I^2 \approx \omega_I^{-2} \oplus \omega_I$, where $\omega_I^{-2} = \mathrm{Hom}(\omega_I^{\otimes 2}, A/I)$. If every projective $A/I$–module splits as a direct sum of a free module and a module of rank one (e.g. dim $A/I = 1$), then every projective $A/I$–module $P$ of rank $r$ is completely determined by $\Lambda^r P$ and hence in this case the surjection $I \twoheadrightarrow \omega_I$ is immediate. This remark and the fact that projective modules over polynomial rings over fields are free (Quillen–Suslin Theorem) led Ferrand–Szpiro to deduce that local complete intersection curves in $\mathbb{A}^3$ are set–theoretic complete intersections. Later, Mohan Kumar (MK1] generalized the Ferrand–Szpiro argument to show that any local complete intersection curve in $\mathbb{A}^n$ is a set–theoretic complete intersection.

We do not know the answer even when $n = 4$ and $V = V(I)$ is a smooth surface and $k$ is algebraically closed.

**Lemma** 1.2. *Let* $R = k[X_1,X_2,X_3,X_4]$, *where* $k$ *is an algebraically closed field and* $I \subset R$ *a local complete intersection ideal of height* 2. *Let* $A = R/I$ *and let* $\omega_I = \omega = \mathrm{Ext}_R^2(A,R)$. *Then*

1) $I/I^2 \approx A \oplus \omega^{-1}$
2) *Consider the following conditions*
   a) $\omega$ *is generated by two elements.*
   b) $I/I^2 \approx \omega \oplus \overline{\omega}^2$
   c) $\omega^{-2}$ *is generated by two elements. We have* a) $\Rightarrow$ b) $\Rightarrow$ c).

**Proof** : 1) Since projective $R$–modules are free and $I$ has projective dimension one, we have an exact sequence $0 \to R^{\ell-1} \to R^{\ell} \to I \to 0$. Tensoring this sequence with $A = R/I$, we get an exact sequence $0 \to L \to A^{\ell-1} \to A^{\ell} \to I/I^2 \to 0$ with $L$ a projective $A$–module of rank one. Thus in $K_0(A)$, we have $[I/I^2] = [A]+[L]$, and hence $L \approx \Lambda^2 I/I^2 = \omega^{-1}$. Since cancellation holds for projectives over $A$ [Su], we have $I/I^2 \simeq A \oplus \omega^{-1}$.

2) a) $\Rightarrow$ b). By (1), $I/I^2 \oplus A \approx A^2 \oplus \omega^{-1}$. Since $\omega$ is generated by two elements, whe have $\omega \oplus \omega^{-1} \approx A^2$. So

$$\begin{aligned} I/I^2 \oplus A &\approx \omega \oplus \omega^{-1} \oplus \omega^{-1} \approx \omega \oplus (\omega^{-1} \otimes A^2) \\ &\approx \omega \oplus \omega^{-1} \otimes (\omega \oplus \omega^{-1}) \approx A \oplus \omega \oplus \omega^{-2}. \end{aligned}$$

Now b) follows from [Su].

3) b) $\Rightarrow$ c). 1) and b) imply that $\omega^2 \oplus \omega^{-1} \approx A \oplus \omega$. Hence $\omega^2 \oplus \omega^{-2} \oplus \omega \approx \omega^2 \oplus A \oplus \omega^{-1} \approx A^2 \oplus \omega$.

Hence by cancelling $\omega$, we have $\omega^2 \oplus \omega^{-2} \approx A^2$, i.e., $\omega^2$ is generated by two elements.