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DETECTING TRIVIAL ELEMENTS OF PERIODIC QUOTIENT OF HYPERBOLIC GROUPS

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Abstract. — In this article we give a sufficient and necessary condition to determine whether an element of the free group induces a nontrivial element of the free Burnside group of sufficiently large odd exponents. Although this result is "well known" among specialists, it has never been stated with such a level of simplicity. Moreover, our proof highlights some important differences between the Delzant-Gromov approach to the Burnside problems and others that exist. This criterion can be stated without any knowledge regarding Burnside groups, in particular about the proof of its infiniteness. Therefore, it also provides a useful tool to study outer automorphisms of Burnside groups. In addition, we state an analogue result for periodic quotients of torsion-free hyperbolic groups.

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RÉSUMÉ (*Un critère pour détecter les éléments triviaux dans un quotient périodique d'un groupe hyperbolique*). — Dans cet article, on propose une condition nécessaire et suffisante pour déterminer si un élément du groupe libre induit ou non un élément trivial dans les groupes de Burnside libre d'exposants impairs suffisamment grands. Bien que ce résultat soit « bien connu » des spécialistes, il n'a jamais été énoncé avec un tel niveau de simplicité. En outre la preuve met en lumière les principales différences entres l'approche de Delzant-Gromov du problème de Burnside et les autres existants dans la littérature. Ce critère peut être énoncé sans aucun pré-requis sur les groupes de Burnside. En particulier il n'est pas nécessaire de comprendre pourquoi les groupes de Burnside sont infinis pour l'appliquer. Pour cette raison il fournit un outil effectif qui nous permettra plus tard d'étudier les automorphismes du groupe de Burnside. Nous donnons aussi un résultat analogue pour les quotients périodiques d'un groupe hyperbolique sans torsion.

Introduction

Let n be an integer. A group G is said to have exponent n if for every $g \in G$, $g^n = 1$. In 1902, W. Burnside asked whether a finitely generated group with a finite exponent is necessarily finite [3]. In order to study this question, the natural object to look at is the free Burnside group of rank r and exponent n denoted by $\mathbf{B}_r(n)$. It is the quotient of the free group $\mathbf{F}_r$ of rank r by the (normal) subgroup $\mathbf{F}_r^n$ generated by the n-th power of all elements of $\mathbf{F}_r$. Therefore, it is the largest group of rank r and exponent n. For a long time, one only knew that for some small exponents, $\mathbf{B}_r(n)$ was finite ($n = 2$ Burnside [3], $n = 3$ Burnside [3] Levi and van der Waerden [16], $n = 4$ Sanov [22], $n = 6$ Hall [14]). In 1968, P.S. Novikov and S.I. Adian achieved a breakthrough. In a series of three papers, they proved that if $r \geq 2$ and $n \geq 4381$ is odd then $\mathbf{B}_r(n)$ is infinite [19]. This result has been improved in many directions. A.Y. Ol'shanskiĭ proposed a different proof of the Novikov-Adian theorem using graded diagrams [20]. S.V. Ivanov [15] and I.G. Lysenok [17] solved the case of even exponents.

The different works cited here develop a similar general strategy. They construct by induction a sequence of groups (G_k) whose direct limit is $\mathbf{B}_r(n)$. Each group is built from the previous one by adjoining new relations. Through a deep study of the properties of these relations, they prove the following key fact. Let p be an integer. Let w be a reduced word of $\mathbf{F}_r$. If w does not contain a subword of the form u^p, then w induces a nontrivial element of $\mathbf{B}_r(n)$ for exponents n very large compared to p (the precise values of p and n depend on the techniques used during the construction). In particular, two distinct reduced words not containing "large power" induce distinct elements in the Burnside group of sufficiently large exponents. The infiniteness of $\mathbf{B}_r(n)$ follows then from the existence of an infinite word without third-power (for instance the Thue-Morse word [23, 24] or other examples [1, Paragraph 3.5]).

More recently, T. Delzant and M. Gromov provided an alternative proof for the infiniteness of Burnside groups [12]. Their work relies on a purely geometrical point of view of small cancellation theory. Using this tool, they also construct a sequence of groups (G_k) whose limit is $\mathbf{B}_r(n)$. However, contrary to the previous approaches, they *do not need* a criterion to distinguish elements in $\mathbf{B}_r(n)$. The infiniteness of Burnside groups indeed follows from the hyperbolic structure of the approximation groups G_k.

One goal of this paper is to highlight the differences and similarities between the Delzant-Gromov approach and other methods. We explain how hyperbolic geometry can be used to recover a sufficient and necessary condition to determine whether an element of a free group induces a trivial element of the free Burnside groups of sufficiently large exponents. Let ξ and n be two integers. An (n, ξ)-*elementary move* consists in replacing a reduced word of the form $pu^m s \in \mathbf{F}_r$ by the reduced representative of $pu^{m-n}s$, provided m is an integer larger than $n/2 - \xi$. Note that an elementary move may increase the length of the word.

THEOREM. — *There exist numbers ξ and n_0 such that for all odd integers $n \geq n_0$ we have the following property. Let w be a reduced word of $\mathbf{F}_r$. The element of $\mathbf{B}_r(n)$ defined by w is trivial if and only if there exists a finite sequence of (n, ξ)-elementary moves that sends w to the empty word.*

This statement is "well known" from the specialists of Burnside groups. To our knowledge, however, it has never been formulated at such a level of simplicity. The reader can, for instance, compare our definition of elementary moves with that of *simple r-reversal of rank α* used by S.I. Adian [1, Paragraph 4.18 and Pages 8-16 for the prerequisites]. Adian's [1] or Ol'shanskiĭ's work [20, Lemma 5.5], as they appear in the literature, would lead to a similar result, but with a *weaker* requirement to perform elementary moves ($m \geq 90$ and $m \geq n/3$, respectively). It may be possible to adapt their techniques to obtain the same theorem. Nevertheless, the purpose of our proof is to emphasize how all this combinatorial machinery can be reinterpreted in terms of stability of quasi-geodesics in an appropriate hyperbolic space.

Before describing our method, we would like to discuss other motivations for this work. We wish to investigate the outer automorphisms of Burnside groups. Since $\mathbf{F}_r^n$ is a characteristic subgroup of $\mathbf{F}_r$, the projection $\mathbf{F}_r \twoheadrightarrow \mathbf{B}_r(n)$ induces a map $\operatorname{Out}(\mathbf{F}_r) \to \operatorname{Out}(\mathbf{B}_r(n))$. This map is neither one-to-one nor onto. Nevertheless, it provides numerous examples of automorphisms of free Burnside groups. For instance, if n is an odd exponent large enough, the image of $\operatorname{Out}(\mathbf{F}_r)$ in $\operatorname{Out}(\mathbf{B}_r(n))$ contains free groups of arbitrary rank [7]. One natural question is: which automorphisms of $\mathbf{F}_r$ induce automorphisms of infinite order of $\mathbf{B}_r(n)$? In [7], the author provided a large class of automorphisms of $\mathbf{F}_r$ with this property. However, we are looking for a sufficient and necessary condition to characterize them. To understand the difficulties that may appear, let us have a look at a simple example that was studied by E.A. Cherepanov [4]. Let $\{a, b\}$ be a free basis of $\mathbf{F}_2$. Let φ be the automorphism of $\mathbf{F}_2$ defined by $\varphi(a) = ab$ and $\varphi(b) = a$. Let us compute the orbit of b under φ.

$$\begin{array}{ll} \varphi^1(b) = a & \varphi^5(b) = abaababa \\ \varphi^2(b) = ab & \varphi^6(b) = abaababaabaab \\ \varphi^3(b) = aba & \varphi^7(b) = abaababaabaababaababa \\ \varphi^4(b) = abaab & \dots \end{array}$$

This sequence converges for the prefix topology to a right-infinite word

$$\varphi^\infty(b) = abaababaabaababaababaabaababaabaab\dots$$

which does not contain a subword that is a fourth-power [18]. Using the criterion of P.S. Novikov and S.I. Adian, the $\varphi^k(b)$'s define pairwise distinct elements of $\mathbf{B}_r(n)$ for some large n. In particular, φ induces an automorphism of infinite order of free Burnside groups of large exponents. For an arbitrary automorphism the situation becomes considerably more complicated. Let $\{a, b, c, d\}$ be a free basis of $\mathbf{F}_4$. Consider the automorphism ψ of $\mathbf{F}_4$ defined by $\psi(a) = a$, $\psi(b) = ba$, $\psi(c) = c^{-1}bcd$ and $\psi(d) = c$. As previously, we compute the orbit of d under ψ.

$$\begin{aligned} \psi^1(d) &= c \\ \psi^2(d) &= c^{-1}\mathbf{b}cd \\ \psi^3(d) &= d^{-1}c^{-1}b^{-1}c\mathbf{ba}c^{-1}bcdc \\ \psi^4(d) &= c^{-1}d^{-1}c^{-1}b^{-1}ca^{-1}b^{-1}c^{-1}bcd\mathbf{ba}^2d^{-1}c^{-1}b^{-1}cbac^{-1}bcdbcd \\ \psi^5(d) &= d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}b^{-1}ca^{-1}b^{-1}c^{-1}bcda^{-2}b^{-1}d^{-1}c^{-1}b^{-1}c\dots \\ &\quad bac^{-1}bcdc\mathbf{ba}^3c^{-1}d^{-1}c^{-1}b^{-1}ca^{-1}b^{-1}c^{-1}bcdba^2d^{-1}c^{-1}b^{-1}c\dots \\ &\quad bac^{-1}bcdcbac^{-1}bcdc \end{aligned}$$

Note that each time $\psi^k(d)$ contains a subword ba^m, then $\psi^{k+1}(d)$ contains ba^{m+1}. In other words, the $\psi^k(d)$'s contain arbitrary large powers of a; hence the previous strategy does not apply. This pathology cannot be avoided by choosing the orbit of an another element. Therefore, we need a more accurate criterion to distinguish two different elements of $\mathbf{B}_r(n)$, which is provided by our main theorem. This question about automorphisms of $\mathbf{B}_r(n)$ is solved in a joint work with Hilion [10].

Another motivation for this work is to understand the exponential growth rate of free Burnside groups. Let G be a group generated by a finite set S. In order to measure its "size" one defines the *(exponential) growth rate* with respect to S by

$$\lambda_G = \lim_{\ell \to +\infty} \sqrt[\ell]{|B(\ell)|},$$

where $|B(\ell)|$ stands for the cardinality of the ball of radius ℓ for the word metric induced by S. The group G is said to have *exponential growth* if $\lambda_G > 1$ (this notion does not depend on the generating set S). For instance, the growth rate of $\mathbf{F}_r$ with respect to a free basis is $2r-1$. Free Burnside groups of large exponents are known to have exponential growth [1, Theorem 2.15]. The author proved that the exponential growth rate of $\mathbf{B}_r(n)$ (with respect to the image of a free basis of $\mathbf{F}_r$) can actually be made arbitrarily close to $2r-1$ [6]. To that end, he bounds from above the difference between the respective growth rate of $\mathbf{F}_r$ and $\mathbf{B}_r(n)$ as a function of r and n. The sharpness of these estimates directly depends on the accuracy of the criterion used to characterize the trivial elements of $\mathbf{B}_r(n)$. Our criterion provides the best known estimate of the growth rate of $\mathbf{B}_r(n)$.

Let us now say a few words about the proof of the main theorem. In [12], T. Delzant and M. Gromov construct a sequence of groups $\mathbf{F}_r \twoheadrightarrow G_1 \twoheadrightarrow G_2 \twoheadrightarrow \dots$ whose direct limit is $\mathbf{B}_r(n)$. At each step the groups G_k have – among others – the following properties.

- G_{k+1} is a small cancellation quotient of G_k
- The relations that define the quotient $G_k \twoheadrightarrow G_{k+1}$ are n-th powers of elements of G_k.

The particularity of the Delzant-Gromov approach is to make *explicit* use of hyperbolic geometry. As a small cancellation quotient, each G_k is indeed a hyperbolic group. Actually, they act on a hyperbolic space X_k whose geometry is finer than the one of the Cayley graph. The geometry of X_k reflects the combinatorial properties of G_k. Let w be a reduced word of $\mathbf{F}_r$. The first space X_0 is just the Cayley graph of $\mathbf{F}_r$. Hence, w can be thought as a path of X_0. If w does not contain large powers (i.e. no large subword of a relation), then it induces a local quasi-geodesic in each X_k. As a consequence of the stability of quasi-geodesics, it cannot loop back to its starting point, thus does not represent a trivial element of $\mathbf{B}_r(n)$. If the word w contains large powers the argument is slightly more complicated. However, the negative curvature of the spaces X_k allows us to understand in a systematic way what prevents w to induce a local quasi-geodesic. To detail this idea let us have a look at the first step $G_0 \to G_1$ of the process.

Given a small cancellation group, one knows an algorithm solving the word problem. Consider for instance w a reduced word of $\mathbf{F}_r$ which is trivial in the first quotient G_1. According to the Greendlinger Lemma, w contains a subword that equals 3/4 of a relation. In our situation, this means that w can be written $w = pu^m s$ where $m \geq 3n/4$. Applying an elementary move, we obtain a new word w', which represents $pu^{m-n}s$ and is shorter than the previous one. Moreover, w' is still trivial in G_1. By iterating the process we get a sequence of elementary moves that sends w to the empty word.

For the Burnside groups the process is more tricky. Let w be a reduced word of $\mathbf{F}_r$, which is trivial in $\mathbf{B}_r(n)$. Since $\mathbf{B}_r(n)$ is the direct limit of the G_ks, there