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TIGHT CLOSURE AND STRONG F-REGULARITY

by Melvin HOCHSTER[1] and Craig HUNEKE[1]

This paper is written in celebration of the contributions of Pierre Samuel to commutative algebra.

1. Introduction.

Throughout this paper all rings are commutative, with identity, and Noetherian, unless otherwise specified. In [HH1] and [HH2] the authors introduced the notion of the tight closure of an ideal and the tight closure of a submodule of a finitely generated module for Noetherian rings which are either of positive prime characteristic p or else are algebras essentially of finite type over a field of characteristic 0. This notion enabled us to give new proofs, which are especially simple in characteristic p, of a number of results (not all of which were perceived to be particularly related) : that rings of invariants of linearly reductive groups acting on regular rings are Cohen–Macaulay, that the integral closure of the n^{th} power of an n generator ideal of a regular ring is contained in the ideal (the Briançon–Skoda theorem), of the monomial conjecture, and of the syzygy theorem. The new proofs yield much more general theorems. For example, we can show by these methods that if S is any Noetherian regular ring containing a field and R is a direct summand of S as an R-module (we shall sometimes say, briefly, that R is a *summand* of S to describe this situation : we always mean $R \longrightarrow S$ is R-split) then R is Cohen–Macaulay. This result was not previously known in this generality. Moreover, this illustrates the general principle that results proved using tight closure techniques but which do not refer specifically to tight closure can be extended to the general equicharacteristic case by using Artin approximation to reduce to a situation in which tight closure is defined.

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One of the most important characteristics of tight closure is that in a regular ring every ideal is tightly closed. We call the Noetherian rings all of whose localizations have this property "F–regular". (The "F" in "F–regular stands for "Frobenius" : the reason for this usage will become clear later). This is an important class of rings which includes the rings of invariants of linearly reductive groups acting on regular rings. A key point is that if S is F–regular and R is a direct summand of S as an R–module then R is F–regular. It turns out that, under mild conditions (like being a homomorphic image of a Cohen–Macaulay ring or a weakening of the requirements for excellence), F–regular rings, which are always normal, are Cohen–Macaulay as well. This is the basis for our new proof that direct summands of regular rings are Cohen–Macaulay in the equicharacteristic case.

Our objectives in this paper are, first, to recap briefly some of the features of tight closure, and then to focus on the notion of a "strongly F–regular" ring. It turns out that rings of invariants of reductive groups have, in fact, this stronger property, and that the stronger property has numerous apparent advantages over F–regularity. We should point out right away that we do not know whether the notions of F–regularity and strong F–regularity are really different in good cases. It would be very worthwhile if it could be proved that the two notions coincide.

2. A survey of tight closure.

Unless otherwise specified A, R, and S denote Noetherian commutative rings with 1. By a "local ring" we always mean a Noetherian ring with a unique maximal ideal. R^0 denotes the complement of the union of the minimal primes of R. I and J always denote ideals. Unless otherwise specified given modules M and N are assumed to be finitely generated.

We make the following notational conventions for discussing "characteristic p". We shall always use p to denote a positive prime integer. We shall use e for a variable element of $\mathbb{N}$, the set of nonnegative integers, and q for a variable element of the set $\{p^e : e \in \mathbb{N}\}$.

If R is reduced of characteristic p we write $R^{1/q}$ for the ring obtained by adjoining all q^{th} roots of elements of R: the inclusion map $R \subseteq R^{1/q}$ is isomorphic with the map $F^e : R \longrightarrow R$, where $q = p^e$, F is the Frobenius endomorphism of R and F^e is the e^{th} iteration of F, i.e. $F^e(r) = r^q$. When R is reduced we write R^∞ for the R–algebra $\cup_q R^{1/q}$. Note that R^∞ is an exception to the rule that the rings we consider be Noetherian.

If $I \subseteq R$ and $q = p^e$ then $I^{[q]}$ denotes $(i^q : i \in I) = F^e(I)R$. If S generates I then $\{i^q : i \in S\}$ generates $I^{[q]}$.

We are now ready to define tight closure for ideals in the characteristic p case.

DEFINITION. *Let $I \subseteq R$ of characteristic p be given. We say that $x \in I^*$, the* <u>*tight closure*</u> *of I, if there exists $c \in R^o$ such that $cx^q \in I^{[q]}$ for all $q \gg 0$*, i.e. *for all sufficiently large q of the form p^e. If $I = I^*$ we say that I is* <u>*tightly closed*</u>.

Remarks. Note that if R is a domain, which is the most important case, the condition that $c \in R^0$ is simply the condition that $c \neq 0$. Note also that if R is reduced then $cx^q \in I^{[q]}$ iff $c^{1/q}x \in IR^{1/q}$. Thus, if $x \in I^*$ then for some $c \in R^0$ we have that $c^{1/q}x \in IR^\infty$ for all q (this condition gets stronger as q gets larger). This gives a heuristic argument for regarding x as being "nearly" in I or, at least, IR^∞: it is multiplied into IR^∞ by elements which, in a formal sense, are getting "closer and closer" to 1 (since $1/q \longrightarrow 0$ as $q \longrightarrow \infty$).

We also note that if R is reduced or if I has positive height it is not hard to show that $x \in I^*$ iff there exists $c \in R^0$ such that $cx^q \in I^{[q]}$ for all q.

We extend this notion to finitely generated algebras over a field of characteristic 0 as follows :

DEFINITION. *Let R be a finitely generated algebra over a field K of characteristic 0, $I \subseteq R$, and $x \in R$. We say that x is in the* <u>*tight closure*</u> *I^* of I if there exist an element $c \in R^0$, a finitely generated $\mathbb{Z}$–subalgebra D of K, a finitely generated D–subalgebra R_D of R containing x and c, and an ideal I_D of R_D such that I_D and R_D/I_D are D–free, the canonical map $K \otimes_D R_D \longrightarrow R$ induced by the inclusions of* K *and R_D in R is a K–algebra isomorphism, $I = I_D R$, and for every maximal ideal m of D, if $\kappa = D/m$ and p denotes the characteristic of κ, then $c_\kappa x_\kappa^q \in I_\kappa^{[q]}$ in $R_\kappa \cong R_D/mR_D$ for every $q = p^e \gg 0$, where the subscript κ denotes images after applying $\kappa \otimes_D$. If $I = I^*$ we say that I is* <u>*tightly closed*</u>.

It is not even completely clear from this definition that I^* is an ideal, although it is not difficult to establish. Our attitude in this survey is as follows : we give a number of proofs in characteristic p to illustrate how easy many arguments are while in characteristic 0 we state results but omit discussion of the proofs (generally speaking, the arguments are rather technical but hold few surprises).

We also note that if R is an algebra essentially of finite type over a field K of characteristic 0, and $I \subseteq R$, we can define the <u>*tight closure*</u>, I^*, of I as $\cup_B (I \cap B)^*$, where the union is extended over all finitely generated K–subalgebras B of R such that R is a localization of B. However, we shall not discuss the situation for algebras essentially of finite type over a field in any detail in this paper.

The next result shows, among other things, how one uses tight closure to prove that direct summands of regular rings are Cohen–Macaulay (C–M).

2.1. THEOREM. *Let R, S denote Noetherian rings which are either of characteristic p or else essentially of finite type over a field.*

a) *If R is regular, every ideal of R is tightly closed.*

b) *If $R \subseteq S$ are domains and J is tightly closed in S then $J \cap R$ is tightly closed in R. (When R and S are not necessarily domains we may assume instead that $R^0 \subseteq S^0$).*

c) *Let $R \subseteq S$ be domains such that every ideal of R is contracted from S (this holds, in particular, if R is a direct summand of S as an R–module). If every ideal of S is tightly closed then every ideal of R is tightly closed.*

d) *The tight closure of an ideal I of R is contained in the integral closure I^- of I.*

e) *If R is a locally unmixed homomorphic image of a C–M ring and $x_1,\ldots,x_n \in R$ have the property that any t of the $x's$ generate an ideal of height $\geq t$, then $(x_1,\ldots,x_{n-1}) :_R x_nR \subseteq (x_1,\ldots,x_{n-1})R^*$, where $I :_R J = \{r \in R : rJ \subset I\}$.*

Sketch of the proof in characteristic p. a) Suppose that $I \subseteq R$, that R is regular, and that $x \in I^*-I$. By localizing at a prime containing $I :_R xR$ we may assume that (R,m) is local as well. If $cx^q \in I^{[q]}$ for all $q > q'$ then $c \in \cap_{q>q'}(I^{[q]}:x^q) = \cap_{q>q'}(I:x)^{[q]}$ (the flatness of the Frobenius endomorphism for regular rings implies that $(I^{[q]}:x^q) = (I:x)^{[q]}) \subseteq \cap_{q>q'} m^q = (0)$, so that $c = 0$.

b) is immediate from the definition of tight closure and c) is immediate from b).

d) (The reader may want to look at the discussion of integral closure given in (2.8) below before going through this argument). We may use a). Suppose $x \in I^*$ and $c \in R^0$ is such that $cx^q \in I^{[q]}$ for all $q \gg 0$. Let $h : R \longrightarrow V$ with $\ker h$ a minimal prime of R, where V is a DVR. Then $h(c)h(x)^q \in (IV)^{[q]}$ for all $q \gg 0$ and $h(c) \neq 0$, and so $h(x) \in (IV)^* = IV$ (since V is regular), and we are done. On the other hand, we may argue directly as follows: Let $I = (x_1,\ldots,x_h)$. Applying the discrete valuation v to the equation $cx^q = \Sigma_{t=1}^h r_{qt}\, x_t^q$ yields $v(c) + qv(x) \geq q \min\{v(x_t):t\}$. Dividing by q and taking the limite as $q \longrightarrow \infty$ yields the result.

e) We shall not prove the result stated in full generality here : we refer the reader to [HH2]. However, we shall give the argument in the special case where the x_i are contained in a regular ring $A \subseteq R$ and R is module–finite over A. In many good cases it is possible to reduce to this case by localizing and completing R and then choosing A properly. In the interesting case (where the x_i do not generate the unit ideal) we may reduce to the situation where A is local,