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# SET–THEORETIC GENERATION OF IDEALS

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(Dedicated to Professor P. Samuel)

SUMMARY

We study the problem of whether a given surface in affine space is a set–theoretic complete intersection. We show, in particular, that surfaces which are birational to a product of curves are set–theoretic complete intersections.

RESUME

On étudie le problème de savoir si une surface donnée dans un espace affine est une intersection complète ensembliste. On démontre en particulier qu'une surface birationellement équivalente à un produit de courbes est une telle intersection.

## §0. Introduction.

In this paper, we study set–theoretic generators of ideals in affine algebras. We will be working over an algebraically closed field $k$. We will prove a sufficient condition for a smooth surface $X$ to be a set–theoretic complete intersection in $\mathbb{A}^n$ $n \geq 5$). This condition is trivially satisfied by a birationally ruled surface. We will show that this condition is satisfied by surfaces birational to product of curves. Spencer Bloch has recently shown to me that this condition is also satisfied by surfaces birational to abelian surfaces.

Another problem we attempt in this article is whether a codimension one subvariety of a smooth affine variety $X$ of dimension $n$ is set–theoretically defined by $n{-}1$ equations. The main interest in this problem, at least for the author, is that if this were not so, then one can find stably trivial non–trivial bundles of rank $n{-}1$ on such varieties. To see why this case is interesting, the reader may see [3]. Of course, the problem is easy when $n{=}1$ or $2$. The real difficulty is from $n{=}3$. We will show that when $n \geq 3$, a subvariety as above is set–theoretically the zeroes of a section of a stably free, rank $n{-}1$ module. For a precise statement, see Theorem 2.

I thank Professors M. Raynaud and L. Szpiro for including me in the Samuel Colloquium. I thank Professor M.P. Murthy for many discussions on the subject matter of this article and Professor Spencer Bloch for showing me how my results apply to the case of surfaces birational to abelian surfaces as well.

## §1. Surfaces.

Let $X \subset \mathbb{A}^n$ be a smooth affine surface. Let $A$ denote the coordinate ring of $X$. Let $P =$ the conormal module of $X$ in $\mathbb{A}^n$.

THEOREM (Boratynski [1]) $X \subset \mathbb{A}^n$ *is a set-theoretic complete intersection if and only if the ideal* $S_+(P) =$ *positively graded elements in* $R = S(P)$, *the symmetric algebra of* $P$ *over* $A$, *is a set-theoretic complete intersection in* $R$.

We say that $A$ satisfies $(\star)$ if for any $z \in A_0(A) =$ zero–cycles modulo rational equivalence, there exists $L_1,\ldots,L_n \in \operatorname{Pic} A$ such that $z = \Sigma_{i=1}^n (L_i.L_i)$, where $(L.L)$ denotes the intersection product in the Chow–ring.

THEOREM 1. *Let $A$ be the co-ordinate ring of a smooth surface. Let $P$ be any $A$-projective module with rang $P \geq 3$. Let $R = S(P)$ = symmetric algebra of $P$ over $A$ and $I = S_+(P)$, the ideal of positively graded elements. If $A$ satisfies $(\star)$, then $I$ is a set-theoretic complete intersection in $R$.*

To prove this theorem, we introduce the notion of modifications. Let the notation be as in the theorem. A projective module $Q$ over $A$ is said to be a modification of $P$, written $Q[P]$, if

i) rank $Q$ = rank $P$,

ii) there exists an $A$–algebra homomorphism $f : S(Q) \longrightarrow S(P)$, such that $\mathrm{rad}(f(S_+Q)) = S_+(P)$.

REMARKS :

i) If $Q_2[Q_1]$ and $Q_3[Q_2]$ then $Q_3[Q_1]$.

ii) If $P \approx Q \oplus L$ where $L \in \mathrm{Pic}\, A$ then $(Q \oplus L^m)[P]$ for any $m \geq 1$.

The first remark is obvious and the second remark follows, once we use the natural map $S(L^m) \longrightarrow S(L)$ for any $m \geq 1$.

PROOF OF THE THEOREM : We need only to show that $P$ can be modified to a free module. Let $L = \det P$. Since $\dim A = 2$ and rank $P \geq 3$, by Serre's theorem [9], there exists a projective module $Q$ such that $P \approx Q \oplus L^{-1}$. Then $\det Q = L^{\otimes 2}$. By remark ii), $Q \oplus L^{-\otimes 2}$ is a modification of $P$. Also $\det(Q \oplus L^{-\otimes 2}) = A$. Thus we may assume that $\det P = A$. Let $c_2(P) \in A_0(A)$ be the second chern class of $P$. $A_0(A)$ is divisible [see e.g. [6], Lemma 2.3]. So we may write $c_2(P) = 3z$. Since $A$ satisfies $(\star)$, we may write $z = \Sigma_{i=1}^{n}(L_i.L_i)$ with $L_i \in \mathrm{Pic}\, A$. Now, the proof is by induction on $n$. If $n = 0$, then $z = 0$ and by [5], $P$ is free.

We will show that $P$ can be modified to a projective module $P'$ with $\det P' = A$ and $c_2(P) = 3z'$, where $z' = \Sigma_{i=1}^{n-1}(L_i.L_i)$. This will complete the proof.

For notational simplicity let $M = L_n$. As before we may write $P = P_1 \oplus M$. Let $c$ denote the total chern class. Then we have

a) $c(p) = c(P_1).(1+c_1(M))$.

By Remark ii), $P_1 \oplus M^{\otimes 2}$ is a modification of $P$. Again we may write $P_1 \oplus M^{\otimes 2} = P_2 \oplus M^{\otimes 1}$. Then we have

b) $c(P_1).(1+2c_1(M)) = c(P_2).(1-c_1(M))$.

Again by Remark ii), $P_2 \oplus M^{\otimes 2}$ is a modification of $P_1 \oplus M^{\otimes 2}$ and hence by Remark i), a modification of $P$. Using a) and b) we may compute $c(P_2 \oplus M^{\otimes 2})$ and then we will get

$$c(P_2 \oplus M^{\otimes 2}) = 2+3z-3(M.M).$$

Thus $P' = P_2 \oplus M^{\otimes 2}$ has all the properties we wanted to achieve. This finishes the proof of the theorem.

COROLLARY 1. (Murthy) *If* $X \subset \mathbb{A}^n$, $X$ *a smooth surface which is birationally ruled, then* $X$ *is a set-theoretic complete intersection.*

PROOF : For $n \leq 4$ see [4].

PROPOSITION. *If* $A$ *is birational to a product of curves then* $A$ *satisfies* $(\star)$.

PROOF : Let $A$ be birational to $C_1 \times C_2$ where $C_i$ are smooth projective curves. We may also assume that $C_i$'s have positive genus ; if not $A$ is birationally ruled and so $A$ satisfies $(\star)$ trivially. Let $Y$ be a smooth projective completion of $X = \operatorname{Spec} A$. Then we have a birational morphism $\pi : Y \longrightarrow C_1 \times C_2$, by uniqueness of minimal models. Let $Z$ denote the union of exceptional curves of $Y$. Then $Z$ is the union of rational curves. So the natural map $A_0(X) \longrightarrow A_0(X-Z)$ is an isomorphism. Also $\operatorname{Pic} X \longrightarrow \operatorname{Pic}(X-Z)$ is a surjection. Thus we need only prove $(\star)$ for $X$ an affine open subset of $C_1 \times C_2$.

Now, since $A_0(X)$ is divisible, we may write any zero cycle $z = 2t$. Also, since $X$ is affine, we may write $t$ as a sum of points of $X$. So it suffices to prove that for any point $p \in X$, $2p = (L.L)$ in $A_0(X)$ where $L \in \operatorname{Pic} X$. Write $p = (p_1,p_2) \in C_1 \times C_2$. Then $M_1 = p_1 \times C_2$ and $M_2 = C_1 \times p_2$ are divisors on $C_1 \times C_2$. $(M_1.M_2) = p$ and $(M_i.M_i) = 0$ for $i = 1,2$ in $A_0(C_1 \times C_2)$. Then $(M_1 \otimes M_2.M_1 \otimes M_2) = 2p$ in $A_0(C_1 \times C_2)$. Restricting $M_1 \otimes M_2$ to $X$, we get the desired result.

COROLLARY 2. *If* $X \subset \mathbb{A}^n$, *is a smooth surface birational to a product of curves then* $X$ *is a set-theoretic complete intersection.*