H. SCHLICHTKRULL

On some series of representations related to symmetric spaces

Mémoires de la S. M. F. 2e série, tome 15 (1984), p. 277-289

<http://www.numdam.org/item?id=MSMF_1984_2_15__277_0>

Société Mathématique de France
2e série, Mémoire n° 15, 1984,p.277-289

ON SOME SERIES OF REPRESENTATIONS RELATED TO SYMMETRIC SPACES.

by

H. Schlichtkrull

In this paper, the series of representations constructed by M. Flensted-Jensen in [3] and [4] are considered. The main results of [8], on lowest K-types and Langlands parameters of the representations of [3] in the equal rank case, are generalized to the other series as well. The representations are identified with subquotients of parabolically induced representations. The parabolic subgroup we use, $P = MAN$, is cuspidal, and moreover, the symmetric space $M/M \cap H$ satisfies the equal rank condition. The inducing representation $\pi \otimes \nu \otimes 1$ of MAN is given by a Flensted-Jensen representation π of M, and thus the determination of Langlands parameters is reduced to Flensted-Jensen representations of M. Further, these results imply unitarity of the representations under certain conditions (see Theorem 4).

Since the proofs of some of our results are rather straightforward generalizations of those of [8], we do not give all the details in these cases, but refer to [8] in stead.

Our results generalize some results of G. Ólafsson [5], [6] (in fact, Theorem 1 and 3 below were obtained before we received [5] and [6]).

The author expresses his gratitude to the organizers of the conference for the invitation to participate.

1. Notation. Let G/H be a semisimple symmetric space with G and H connected and linear. Let τ be the corresponding involution, and let θ be a commuting Cartan involution. Denote by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding decompositions of the Lie algebra $\mathfrak{g}$, and let K be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Let G_0 denote the analytic subgroup of G with Lie algebra $\mathfrak{g}_0 = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}$.

Choose a θ-invariant maximal abelian subspace $\mathfrak{a}^0$ of $\mathfrak{q}$, and put $\mathfrak{t} = \mathfrak{a}^0 \cap \mathfrak{k}$. Let $\Delta \subset \mathfrak{a}^{0*}_{\mathbb{C}}$ be the set of roots of $\mathfrak{a}^0$ in $\mathfrak{g}_{\mathbb{C}}$, and choose a positive system Δ^+ which is θ-compatible, i.e. $\alpha \in \Delta^+$ and $\alpha|_{\mathfrak{t}} \neq 0$ implies $\theta\alpha \in \Delta^+$. Put $\rho = \rho(\Delta^+) = \frac{1}{2}\sum_{\alpha\in\Delta^+}(\dim \mathfrak{g}^{\alpha}_{\mathbb{C}})\alpha \in \mathfrak{a}^{0*}_{\mathbb{C}}$.

Let $\ell = \mathfrak{g}^{\mathfrak{t}}$ be the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$, and let $\bar{\ell}$ denote the orthocomplement of $\mathfrak{t}$ in ℓ (w.r.t. the Killing form of $\mathfrak{g}$). Choose $\mathfrak{t}_2$ maximal abelian in $\bar{\ell} \cap \mathfrak{k} \cap \mathfrak{q}$, then $\tilde{\mathfrak{t}} = \mathfrak{t} + \mathfrak{t}_2$ is maximal abelian in $\mathfrak{k} \cap \mathfrak{q}$. Let $\Delta_c = \Delta(\tilde{\mathfrak{t}}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$, $\Delta_{c,1} = \{\alpha \in \Delta_c \mid \alpha|_{\mathfrak{t}} \neq 0\}$ and $\Delta_{c,2} = \{\alpha \in \Delta_c \mid \alpha|_{\mathfrak{t}} = 0\}$. Put $\Delta^+_{c,1} = \{\alpha \in \Delta_c \mid \exists\beta \in \Delta^+ : \beta|_{\mathfrak{t}} = \alpha|_{\mathfrak{t}}\}$ and choose a positive system $\Delta^+_{c,2}$ for the root system $\Delta_{c,2}$, then $\Delta^+_c = \Delta^+_{c,1} \cup \Delta^+_{c,2}$ is a positive system for Δ_c. Define $\rho_c = \rho(\Delta^+_c) = \frac{1}{2}\sum_{\alpha\in\Delta^+_c}(\dim \mathfrak{k}^{\alpha}_{\mathbb{C}})\alpha \in i\tilde{\mathfrak{t}}^*$ and $\rho_{c,1} = \rho(\Delta^+_{c,1})$ similarly. Notice that $\rho_{c,1}|_{\mathfrak{t}_2}$ does not vanish in general, but at least we have:

Lemma 1. $\langle\rho_{c,1},\alpha\rangle = 0$ *for all* $\alpha \in \Delta_{c,2}$.

Proof: Let $\alpha \in \Delta_{c,2}$, and denote by s_α reflection in α. Then $s_\alpha(\Delta^+_{c,1}) = \Delta^+_{c,1}$ and hence the lemma. □

For each $\lambda \in \mathfrak{a}^{0*}_{\mathbb{C}}$ we define $\mu_\lambda \in \tilde{\mathfrak{t}}^*_{\mathbb{C}}$ by the following equations:

$$(1) \qquad (\mu_\lambda + 2\rho_c)|_{\mathfrak{t}} = (\lambda+\rho)|_{\mathfrak{t}} \quad \text{and} \quad (\mu_\lambda + 2\rho_{c,1})|_{\mathfrak{t}_2} = 0 .$$

2. Flensted-Jensen's representations. Let $c \geq 0$ be the smallest possible constant such that [4] Theorem 1 holds, and define $\Lambda \subset \mathfrak{a}^{0*}_{\mathbb{C}}$ to be the set of those $\lambda \in \mathfrak{a}^{0*}_{\mathbb{C}}$ satisfying the following conditions (2) and (3):

(2) $\mathrm{Re}\langle\lambda,\alpha\rangle > c$ for all $\alpha \in \Delta^+$ with $\alpha|_{\mathfrak{t}} = 0$

(3) $$\begin{cases} \dfrac{\langle\mu_\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}^+ & \text{for all } \alpha \in \Delta_c^+ \\ \mu_\lambda(X) \in \mathbb{Z} & \text{for } X \in \mathfrak{t},\ \exp 2\pi i X = e\,. \end{cases}$$

For each $\lambda \in \Lambda$ Flensted-Jensen [4] defines a function $\psi_\lambda \in C^\infty(G/H)$ by an integral formula (for the dual function on the dual symmetric space G^0/H^0), and the following properties hold for these functions:

a) The representation of K generated by ψ_λ is finite dimensional and irreducible. Denoting by δ_λ the contragredient of this representation of K, δ_λ is spherical for $K/K\cap H$ and has highest weight μ_λ.

(We have not included Condition (9) of [4], since it is redundant by Lemma 1).

b) ψ_λ is a joint eigenfunction for $U(\mathfrak{g})^K$ acting on $C^\infty(G/H)$ from the left. The eigenvalues are determined as follows: There is a unique homomorphism $\gamma\colon U(\mathfrak{g})^K \to U(\mathfrak{a}^0)$ such that for $u \in U(\mathfrak{g})^K$:

(4) $$u - \gamma(u) \in (\bar{\mathfrak{l}}\cap\mathfrak{k})_{\mathbb{C}}\, U(\mathfrak{g}) + U(\mathfrak{g})\,(\mathfrak{h}_{\mathbb{C}}^{\mathfrak{a}^0} + \mathfrak{n}^0)$$

where $\mathfrak{n}^0 = \sum_{\alpha\in\Delta^+} \mathfrak{g}_{\mathbb{C}}^{\alpha}$. Then $u\psi_\lambda = \gamma(u)(-\lambda-\rho)\psi_\lambda$.

<u>Remark</u>. In the sequel we use only properties a) and b) of the functions ψ_λ. If ψ_λ can be defined (e.g. by analytic continuation in λ), such that a) and b) still hold for some λ which does not satisfy (2), then our results can be extended to these parameters as well.

From a) and b) it follows by [2] Proposition 9.1.10 (iii) that the K-type $\mu_\lambda^\vee$ has multiplicity one in the $\mathfrak{g}$-module generated by ψ_λ. Consequently, this module has a unique irreducible quotient T^λ which contains $\mu_\lambda^\vee$.

If $\mathfrak{t}$ is maximal abelian in $\mathfrak{k}\cap\mathfrak{q}$, then ψ_λ is the same as the function defined in [3]. In this case $c = 0$, but (2) is not necessary for defining ψ_λ. In fact (2) is not serious since one can prove that then $\psi_{s\lambda} = \psi_\lambda$ for all elements s from the Weyl group of the root system $\{\alpha \in \Delta \mid \alpha|_{\mathfrak{t}} = 0\}$. The series of $(\mathfrak{g},K)$-

modules T^λ is in this case called the <u>fundamental</u> <u>series</u> for the symmetric space G/H.

If we can choose $\mathfrak{a}^0$ such that $\mathfrak{t} = \mathfrak{a}^0$, we say that G/H satisfies the <u>equal</u> <u>rank</u> condition. If furthermore $\langle\lambda,\alpha\rangle > 0$ for all $\alpha \in \Delta^+$, then ψ_λ is square integrable with respect to invariant measure on G/H, and hence ψ_λ generates a unitary irreducible representation $\pi_\lambda{}^G$ of G, whose Harish-Chandra module is T^λ. This was proved under stronger assumptions on λ in [3], and subsequently proved in general by T. Oshima (unpublished, cf. however [10] and [13]).

<u>3. Lowest K-types</u>. Let $L = G^{\mathfrak{t}}$, then L is connected and has Lie algebra ℓ. Put $n_1 = \Sigma_{\alpha\in\Delta^+,\alpha|_{\mathfrak{t}}\neq 0}\, \mathfrak{g}_{\mathbb{C}}^\alpha$ and $n_2 = \Sigma_{\alpha\in\Delta^+,\alpha|_{\mathfrak{t}}=0}\, \mathfrak{g}_{\mathbb{C}}^\alpha$, and observe that $\ell_{\mathbb{C}} + n_1$ is a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Choose an Iwasawa decomposition $\ell = \ell \cap \mathfrak{k} \oplus \mathfrak{a} \oplus n_\ell$ such that $\mathfrak{a}^0 \cap \mathfrak{p} \subset \mathfrak{a}$ and $n_2 \subset n_\ell$. Notice that $\mathfrak{a}$ is τ-stable, and $\mathfrak{a} \cap \mathfrak{q} = \mathfrak{a}^0 \cap \mathfrak{p}$ by maximality of $\mathfrak{a}^0$ in $\mathfrak{q}$ so that $\mathfrak{a} = \mathfrak{a}^0 \cap \mathfrak{p} + \mathfrak{a} \cap \mathfrak{h}$. Define $\rho_\ell \in \mathfrak{a}^*$ by $\rho_\ell = \frac{1}{2}\,\mathrm{Tr}\,\mathrm{ad}_{n_\ell}$, then it follows easily that $\rho_\ell|_{\mathfrak{a}\cap\mathfrak{q}} = \rho|_{\mathfrak{a}^0\cap\mathfrak{p}}$. Define for each $\lambda \in \mathfrak{a}_{\mathbb{C}}^{0*}$ an element $\nu_\lambda^L \in \mathfrak{a}_{\mathbb{C}}^*$ by

(5) $$\nu_\lambda^L|_{\mathfrak{a}\cap\mathfrak{q}} = -\lambda|_{\mathfrak{a}^0\cap\mathfrak{p}} \quad \text{and} \quad \nu_\lambda^L|_{\mathfrak{a}\cap\mathfrak{h}} = \rho_\ell|_{\mathfrak{a}\cap\mathfrak{h}}\,.$$

<u>Theorem 1</u>. *Assume* $\lambda \in \Lambda$ *and*

(6) $$\langle(\lambda+\rho)|_{\mathfrak{t}},\ \alpha|_{\mathfrak{t}}\rangle \geq 0 \quad \textit{for all} \quad \alpha \in \Delta^+\,.$$

Then $\mu_\lambda^\vee$ *is a lowest* K-*type of* T^λ, *and* T^λ *has no other lowest* K-*types.*

<u>Proof</u>: Let $\bar{V}_\lambda$ denote the spherical representation of $\bar{L}$ (the analytic subgroup with Lie algebra $\bar{\ell}$) with parameter $\nu_\lambda^L \in \mathfrak{a}_{\mathbb{C}}^*$, and denote by V_λ the representation of L which extends $\bar{V}_\lambda$ with the character $e^{\mu_\lambda - 2\rho(n_1\cap\mathfrak{p})}$ on $\exp i\tilde{\mathfrak{t}}$ (then V_λ is well defined, cf. [8] Lemma 5.5 and the succeeding remark).

Let $X(\ell_{\mathbb{C}}+n_1, V_\lambda, \mu_\lambda)$ be the $(\mathfrak{g},K)$-module induced from V_λ in the sense of [11], then one can conclude by comparing actions of $U(\mathfrak{g})^K$ on μ_λ that the module $T^{\lambda\vee}$, contragradient to T^λ, is equivalent to $X(\ell_{\mathbb{C}}+n_1, V_\lambda, \mu_\lambda)$, (cf. [8] Lemma 5.6 where T^λ has been interchanged with $T^{\lambda\vee}$).