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SOME RESULTS ABOUT EXPONENTIAL FIELDS (SURVEY)

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Summary
In the present paper, a survey about some results from the theory of exponential fields is given. The investigations are motivated by Tarski's decidability problem of the field of real numbers with an additional exponential function. For solving Tarski's problem it seems to be useful to have more information about special exponential fields and classes of such structures. So different axiomatic classes of exponential fields and their theories are investigated. Especially, the solution of the "dominance problem" and the "problem of the last root" for exponential terms are given here.

§ 1 Introduction

In the present paper, a survey about some results from the theory of exponential fields is given. The investigations of this theory are motivated by A. Tarski's decidability problem of the field of real numbers with an additional exponential function. In recent years several people have been concerned with exponential fields and rings and obtained interesting results (see e.g. [Dr], [HR], [M], [R], [Wi], [DW1], [DW2], [Da], [Wo]), but Tarski's problem is still open and a solution is not in sight for the time being. However independent of the mentioned problem, the class of exponential fields is a very interesting subject of investigation. Only the interplay of analytical and algebraic means yields fundamental results, where the algebraic methods have often to be developed first.

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In the papers [DW1], [DW2], [Da], [Wo] B.I. Dahn and I investigated different classes of exponential fields with the intention to get more information on such structures and classes and their theories in order to give perhaps a contribution to the solution of Tarski's decidability problem. The most important results from our papers are presented here in a survey and without proofs.

Definition. If F is a field and E a unary function from F into F, then (F,E) is said to be an exponential field if for all $x,y \in F$ it holds that $E(x+y) = E(x)E(y)$ and $E(o) = 1$, $E(1) \neq 1$. In this case E is said to be an exponential function on F.

In the following let L be a language for exponential fields, i.e. L contains the usual symbols $+, -, \cdot, ^{-1}$ and an additional unary function symbol E for an exponential function. Further, let E_{ax} be the set of axioms $E(x+y) = E(x)E(y)$, $E(o) = 1$, $E(1) \neq 1$ and let EF be an $\forall$-axiom system for fields of characteristic o augmented by E_{ax}. Then EF determines the theory of exponential fields. The most important models of EF are (R,e) and (C,e), where R and C are the fields of real and complex numbers, respectively, and e is the usual exponential function in these fields.
We could also regard exponential fields of characteristic p, p a
$1 = E(o) = E(px) = E(x)^p$ and finally we get $E(x) = 1$ for all x.
in the set of the p-th roots of 1.
In the following Q denotes the field of rational numbers, Z the set of integers, F an arbitrary field of characteristic o and unless stated otherwise m,n,k,l,i,j denote natural numbers.
i can also be $\sqrt{-1}$, the actual meaning of i will be clear from the context. If F is an ordered field and $a,b \in F$, then $|a|$ is the absolute value of a and $a \sim b$ means that $|a-b|$ is smaller than all positive rational numbers. Notions and denotations not specially explained in this paper are used as usual.
Our aim is now to give a contribution to finding a recursive and complete axiom system of $Th(R,e)$ if such a system exists. So we try to approximate this theory by appropriate and natural axioms.

§ 2 Unordered exponential fields

First of all we want to provide some easy, well-known facts.

Fact 1. In (F,E) E is not uniquely determined by F and EF. Indeed, if f is an additive function from F into F and $E(f(1)) \neq 1$, then $E^*(x) = E(f(x))$ is an exponential function on F, too.

Fact 2. (C,e) is strongly undecidable.

The field of rationals is definable in (C,e) by the formula $\varphi(x) := \exists y \exists z (E(y) = E(z) = 1 \wedge z \neq o \wedge x = y/z)$. In fact, $(C,e) \models e^y = 1$ iff $y = 2q\pi i$, where $q \in Z$ and $i = \sqrt{-1}$. Since Q is strongly undecidable (see e.g. [Sh]), we have the claim and, moreover, we obtain

Fact 3. EF is undecidable.

The next lemma shows that the range of the exponential function in every EF-existentially complete model is the whole field, excepting o.

Lemma 4. [DW1]

Let $\mathbf{F} = (F,E)$.

(i). If $a \in F$ and $a \neq o$, then there is an extension $\mathbf{F}^* = (F^*,E^*)$ of $\mathbf{F}$ such that $\mathbf{F}^* \models EF$ and $\mathbf{F}^* \models \exists x(E^*(x) = a)$.

(ii). If $\mathbf{F}$ is EF-existentially complete, then $\mathbf{F} \models \exists x(E(x) = a)$ for all $a \in F$, $a \neq o$.

Similar as for (C,e), there exists a formula $\Psi(x)$ which defines the field of rationals in all EF-existentially complete models.

Theorem 5. [DW1]

Let $\mathbf{F} = (F,E)$ be EF-existentially complete. Then, for all $a \in F$, $a \notin Q$ iff $\mathbf{F} \models \exists x(E(x) = 1 \wedge E(ax) = 2) := \neg\Psi(a)$.

Since $\Psi(x)$ does not define Q in (C,e), we get

Corollary 6. [DW1]

(C,e) is not existentially complete.

By compactness arguments and the strong undecidability of Q we finally obtain from the above theorem:

Corollary 7. [DW1]

(i). EF is not companionable (and hence EF has no model completion).

(ii). Every existentially complete exponential field is strongly undecidable.

Theorem 8. [DW1]

(R,e) has no existential closure, i.e. there is no EF-existentially complete extension of (R,e) that is embeddable in every existentially complete extension of (R,e).

Our results show that the theory of EF is rather complicated and since EF has models with quite different properties, EF is not a good approximation of Th(R,e). Therefore, in the following, we confine ourselves to more special classes of such fields, namely to ordered exponential fields.

§ 3 Ordered exponential fields

Now we are going to study some parts of the universal theory of the ordered field of real numbers with exponentiation.
Let OF be an $\forall$-axiom system for ordered fields and
$T = OF \cup E_{ax} \cup \{(1 + 1/n)^n \leq E(1) \leq (1 + 1/n)^{n+1} : n > o\}$.
Since the statement $\forall x > o \, \forall y (E(y) = 1 + 1/x \longrightarrow E(xy) < E(1))$ is true in (R,e) but not in some non-archimedean T-models, the $\forall$-theory of T is weaker than $Th_\forall(R,e)$.
Hence we regard the better approximation
$OEF = OF \cup E_{ax} \cup \{E(x) \geq 1 + x\}$.
The following theorem, which can be proved by standard arguments, shows that the theory of ordered exponential fields OEF is sufficiently strong to characterize the exponential function uniquely in the standard model (R,e).

Theorem 9. [DW1]

In OEF the following formulas can be proved.

(i). $E(o) = 1$, $E(x) \geq o$.

(ii). $x \neq o \longrightarrow E(x) > 1 + x$, and hence E is strictly monotonously increasing.