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THE COHOMOLOGY OF MONSKY AND WASHNITZER

Dedicated to B. Dwork on the occasion
of his 60^{th} anniversary

by

Marius van der Put

Summary

The Zeta-function of an algebraic variety over a finite field can be expressed in terms of a Frobenius operator acting on p-adic cohomology groups of this variety. Those cohomology groups, based on work of B. Dwork, are called the Monsky-Washnitzer cohomology. The first four sections of this paper give a survey of the papers of Monsky and Washnitzer. Their work is simplified and slightly extended by the use af Artin-approximation and some rigid analysis. In section 5 the connection with Dwork's work is indicated, Adolphson's index theorem is given in a different form in section 6. Dwork's remarkable formula for the unit root of an elliptic curve and properties of the solutions of the hypergeometric differential equation with parameters $\frac{1}{2}$, $\frac{1}{2}$, 1 are proved in detail in section 7.

Résumé

La fonction zêta d'une variété algébrique sur un corps fini peut s'exprimer à l'aide des opérateurs de Frobenius sur des groupes de cohomologie p-adique de cette variété. Ces groupes de cohomologie, qui sont inspirés par des travaux de Dwork, s'appèlent la cohomologie de Monsky et Washnitzer. Les quatre premiers paragraphes développent cette théorie. L'exposé simplifie les papiers de Monsky et Washnitzer grace à une approximation d'Artin et un peu d'analyse rigide. Le paragraphe 5 indique le rapport avec les travaux de Dwork. Un théorème d'indice due à Adolphson est donné dans une forme plus générale dans le paragraphe 6. La formule remarquable de Dwork pour le "unit root" d'une courbe elliptique ainsi

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que des propriétés de l'équation hypergéométrique à paramètres $\frac{1}{2}, \frac{1}{2}$, 1 sont montrés en détail dans le paragraphe 7.

§ 1. Introduction.

The aim of the Monsky-Washnitzer cohomology, based on and inspired by the work of B. Dwork, is to find and explicit expression for the Zeta-function of an algebraic variety X over a finite field $k = \mathbb{F}_q$.

(1.1) $Z(X|k;t) = \exp(\sum_{s \geq 1} \frac{N_s}{s} t_s)$ is this Zeta-function and N_s denotes the number of points of X with valus in $\mathbb{F}_q s$.

Let R denote a complete discrete valuation ring with $R/\pi R = k$ and $K = Qt(R)$ of characteristic 0. (e.g. $R = W(k)$). One tries to find cohomology groups $H^i(X;K)$ (vectorspaces over K) with an induced action F_* on it, coming from the Frobenius map $x \longmapsto x^q$ on X, such that:

(1.2) $N_s = \Sigma (-1)^i Tr((q^n F_*^{-1})^s | H^i(X;K))$ (Lefschetz' fixed point formula)

(1.3) $Z(X\ k;t) = \prod_{i\ odd} P_i(t) \prod_{i\ even} P_i(t)^{-1}$ where $P_i(t) = \det(1 - tq^n F_*^{-1} | H^i(X;K))$.

The papers of MW [11, 12] are mainly concerned with the case: X an affine, regular variety of dimension n. As we will see, this implies that $H^i(X;K) = 0$ for $i > n$. If one knows that $\dim H^i(X;K) < \infty$ for all i, then (1.3) is an easy consequence of (1.2). Moreover Z is clearly a rational function in this case. However, the authors MW have not shown that the $H^i(X;K)$ are finite dimensional. They work instead with nuclear operators L on a vectorspace M over K. The definition can be given as follows: An eigenvalue of L is a $\lambda \in \overline{K}$ = the algebraic closure of K, such that the minimum polynomial g of λ has the property $\ker(g(L)) \neq 0$.

A K-linear map $L: M \longrightarrow M$ is called nuclear if:

(i) For every eigenvalue $\lambda \neq 0$ there exists a decomposition $M = A \oplus B$ with A,B vectorspaces invariant under L; $B = \bigcup_{n \geq 1} \ker(g(L)^n)$ is finite-dimensional and g(L) is bijective on A.

(ii) The nonzero eigenvalues of L, form a finite set or a sequence with limit 0.

B above is the generalized eigenspace by λ and A equals $\bigcap_{n \geq 1} \mathrm{im}(g(L)^n)$. For

$n \in \mathbb{N}$ we denote by M_n the sum of the generalized eigenspaces of L with eigenvalues λ, $|\lambda| \geq |\pi|^n$. Then $\dim M_n < \infty$. Define now $Tr(L^s) = \lim_{n \to \infty} Tr(L^s|M_n)$ and $\det(1-tL) = \lim_{n \to \infty} \det(1-tL|M_n)$. The limits exist and $\det(1-tL)$ is an entire function on K. Moreover

$$(1.4)\quad \det(1-tL) = \exp\left(-\sum_{s \geq 1} \frac{Tr(L^s)}{s} t^s\right)$$

MW prove that $q^n F_*^{-1}$ is nuclear. So (1.2) implies (1.3) and $Z(X|k;t)$ is a meromorphic function on all of K. The power series $Z(X|k;t)$ is also convergent w.r.t. the archimedian valuation on Q. A criterium of Dwork-Borel then shows that $Z(X|k;t)$ is actually a rational function.

We note the following property of nuclear operators: Let $L_i: M_i \longrightarrow M_i$ $(i=1,2)$ be nuclear, let the linear map $\alpha: M_1 \longrightarrow M_2$ satisfy $\alpha L_1 = L_2 \alpha$, then the induces maps L_0 on $\ker \alpha$ and L_3 on $\operatorname{coker} \alpha$ are nuclear. Moreover:

$$(1.5)\quad \prod_{i=0}^{3} \det(1-tL_i)^{(-1)^i} = 1 \text{ and } \sum_{i=0}^{3} Tr(L_i^s) = 0.$$

§ 2. Definition of the Monsky-Washnitzer cohomology.

Let X be a smooth affine variety over $k = \mathbb{F}_q$ with coordinate ring $\bar{A}$. According to a result of M[me] Elkik [15] there exists a R-algebra B, finitely generated and smooth over R such that $B/\pi B \simeq \bar{A}$.

Write $B = R[t_1,\ldots,t_n]/(f_1,\ldots,f_m)$. One replaces B by the ring

$A = R<t_1,\ldots,t_n>^{\dagger}/(f_1,\ldots,f_m)$, where $R<t_1,\ldots,t_n>^{\dagger}$ consists of the power

series $\Sigma a_\alpha t^\alpha$ such that all $a_\alpha \in R$ and for some $C>0$ and $\rho, 0<\rho<1$, one has $|a_\alpha| \leq C\rho^{|\alpha|}$ for all α.

The elements of $R<t_1,\ldots,t_n>^{\dagger}$ are called overconvergent power series. Every element converges in a polydisc $\{(t_1,\ldots,t_n) \in K^n \,|\, |t_1| \leq \rho_1,\ldots,|t_n| \leq \rho_n\}$ with all $\rho_i > 1$.

The ring A satisfies $A/\pi A = \bar{A}$ and A is complete in some weak sense. For our purposes we make the following simplifying definition.

(2.1) Definition. A weakly complete finitely generated (w.c.f.g) algebra over R is a homomorphic image of some $R<x_1,\ldots,x_n>^{\dagger}$.

(2.2) Proposition. $R<x_1,\ldots,x_n>^\dagger$ satisfies Weierstrass' preparation and division.

The proof of (2.2) contains no surprises. Among the consequences are: $R<X_1,\ldots,X_n>^\dagger$ noetherean and $R[X_1,\ldots,X_n] \to R<X_1,\ldots,X_n>^\dagger$ is flat.

(2.3) For $A = R<t_1,\ldots,t_n>^\dagger/(f_1,\ldots,f_m)$ one defines a module of differentials

$$D^1(A) = Adt_1 + \ldots + Adt_n / \text{the A-submodule generated by } \{\frac{\partial f_i}{\partial t_1} dt_1 + \ldots + \frac{\partial f_i}{\partial t_n} dt_n \,|\, i = 1,\ldots m\}$$

This module is the universal finite module of differentials of A/R. It does not depend on the chosen representation of A. It is easily seen that $D^1(A) \otimes \bar{A} = \Omega^1_{\bar{A}/k}$. The module $\Omega^1_{\bar{A}/k}$ is projective and its rank is equal to the dimension d of $\bar{A}$. Using flatness one can conclude that $D^1(A)$ is also projective of rank d. An easier argument uses the Jacobian-criterion. Let I be the ideal in A generated by the $(n-d) \times (n-d)$-minors of $(\frac{\partial f}{\partial t})$. Then $(\pi, I) = A$ since $\bar{A}/k$ is regular of dimension d. Hence I contains an element of the form $(1 - \pi a)$ with $a \in A$. The infinite series $1 + \pi a + \pi^2 a^2 + \ldots$ converges in A and so $1 \in I$. This implies that $D^1(A)$ is a projective module of rank d over A.

As usual one makes the de Rham-complex D(A):

$0 \to D^0(A) \xrightarrow{d^0} D^1(A) \xrightarrow{d^1} D^2(A) \to \ldots$ with $D^i(A) = \Lambda^i D^1(a)$ and d^i = the exterior differentiation. The i^{th}-cohomology group of the complex D(A) is denoted by $H^i(X;R)$ or $H^i(\bar{A}/R)$. Further $H^i(X;K) := H^i(\bar{A}/K) := H^i(\bar{A}/R) \otimes_R K$ is the definition of the Monsky-Washuitzer cohomology. The notations are justified in (2.4).

(2.4) Unicity and the lifting of the Frobenius map.

This section contains some new results. In particular the technical assumption "very smooth" in the MW-papers is removed with the help of a special case of Artin-approximation.

We write $R<t_1,\ldots,t_n>$ for the ring of power series $\Sigma a_\alpha t^\alpha$ with $a_\alpha \in R$ and $\lim a_\alpha = 0$. Clearly $R<t_1,\ldots,t_n>$ is the π-adic completion of $R<t_1,\ldots,t_n>^\dagger$. For any w.c.f.g. algebra A we write $\hat{A} = \varprojlim A/\pi^n A$ for its π-adic completion.

(2.4.1) Proposition. $R<t_1,\ldots,t_n>^\dagger$ has the Artin-approximation property.

The statement means the following: "Let $f_1,\ldots,f_m$ belong to $R<X_1,\ldots,X_a,Y_1,\ldots,Y_b>^\dagger$, let $\varepsilon > 0$ and let $\hat{y}_1,\ldots,\hat{y}_b \in R<X_1,\ldots,X_a>$ satisfy $f_i(X_1,\ldots,X_a,\hat{y}_1,\ldots,\hat{y}_b) = 0$ for $i = i,\ldots,m$. There are $y_1,\ldots,y_b \in R<X_1,\ldots,X_a>^\dagger$