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ABSTRACT : We define an invariant associated to a homomorphism of free modules and show, first, that this generalizes the multiplicity in the sense of Samuel and, second, that in the situation we are considering, the local Chern character of a perfect complex can be defined in terms of this invariant. Some questions are raised as to the positivity of these numbers and connections with mixed multiplicities are described.

One of the common methods in studying ideals and modules over a commutative ring has been to define numerical invariants which reflect their properties. In this paper we look at a few of these invariants, which have been defined in various contexts, and describe some relations between them. Let A be a local ring with maximal ideal $\mathfrak{m}$, and let I be an ideal of A primary to the maximal ideal, so that A/I is a module of finite length. This length is the simplest invariant associated to the ideal, and it could be considered to be the most important one, but Samuel [7] defined a somewhat more complicated one, called the multiplicity of I, and showed that it was often more fundamental in studying both algebraic and geometric questions; since then, of course, this has become a standard part of Commutative Algebra.

The comparison of invariants we discuss in this paper is analogous to the comparison of length and multiplicity of an $\mathfrak{m}$–primary ideal. Take now a bounded complex of free A–modules, which we denote F_*. In place of the assumption that I be primary, we assume that the homology of F_* is of finite length. Again, there are two invariants one can associate to F_*. The first is the Euler characteristic, denoted $\chi(F_*)$, which is the alternating sum of lengths of the homology modules. The second was defined by Baum, Fulton and MacPherson and is defined in terms of the local Chern character. This theory has been extended by Fulton [2], and certain applications have made it appear that here also this more complicated invariant may be more fundamental in studying homological questions in Commutative Algebra (see Roberts [5] [6]).

We give here an alternative construction of this invariant. More precisely, we define an invariant of a map of free modules (or of a matrix, if one chooses to look at it that way) with certain properties (corresponding to finite length, specified below). On the one hand, if this map goes to a rank one free module, the image is a primary ideal, and this is the multiplicity of Samuel. On the other hand, the alternating sum of these is the local Chern character in the second example. We define this, which we call the *multiplicity* of the homomorphism, in section 1, and, in the process, we show that the connection with multiplicities is more that simply an analogy, since the definition itself is in terms of the so–called *mixed multiplicities* of ideals of minors of the matrix. In section 2 we show that it does agree with the other invariants mentioned above. In the third section we consider homomorphisms which can be put into a complex of length equal to the dimension of the ring with homology of finite length and ask some questions concerning the properties of this invariant in that case. Finally, in the last section, we work out a couple of special cases to explain how one step of the construction works in practice.

We remark that one motivation behind this work was to investigate the contributions of the individual boundary homomorphisms of a perfect complex (i.e. a bounded complex of free modules) to the local Chern character. The fact that a complex can be divided up in this way was proven in a construction of Fulton ([2], ex. 18.3.12) to prove his local Riemann–Roch theorem. The construction we give here carries this out explicitly, specifies which locally free sheaves occur in the decomposition in terms of determinants, and gives a formula for each contribution in terms of mixed multiplicities. In addition, it is applied to an independent map of free modules, so that, in particular, it is defined whether the map fits into a perfect complex or not. What this number means when the map does not fit into a perfect complex is not clear, but it is interesting that an invariant like this can be defined in this generality.

1. The multiplicity of a homomorphism of free modules.

Let A be a local ring of dimension d and maximal ideal $\mathfrak{m}$, and let $\phi : E \longrightarrow F$ be a homomorphism of free A–modules. We wish to assume that ϕ is generically of constant rank, and, to simplify the situation here, we assume that A is an integral domain. Let r be the generic rank of ϕ. We define the support of ϕ to be the set of prime ideals of A for which the localization at P is not split of rank r, by which we mean that it is not of the form

$$A^s \otimes A^r \longrightarrow A^r \otimes A^t$$

where the map is $\begin{pmatrix} 0 & 1_{A^r} \\ 0 & 0 \end{pmatrix}$. Let e denote the rank of E and f the rank of F. We assume that the support of ϕ is the maximal ideal of A. We wish to define a number associated to ϕ which satisfy the properties outlined in the introduction.

Let M denote the matrix which defines ϕ. We assume that the bases are chosen so that both the first r rows and the first r columns of M have rank r.

We first define two sequences of ideals associated to the matrix M. We note that these are not canonically defined by the map itself, but depend on the bases chosen for E and F (or, more precisely, on filtrations by free direct summands defined by them). First, for $k = 0,1,...,r$ we let $\mathfrak{e}_k$ denote the ideal generated by the k by k minors of the first k rows of M (for $k = 0$ this is defined to be the unit ideal, i.e. A itself; we include this to avoid special cases in later notation). Next, for $k = 0,1,...,r$ we let $\mathfrak{f}_k$ denote the ideal generated by the r by r minors of the first r columns of M which include the first k rows. Note that these ideals are not necessarily $\mathfrak{m}$–primary. We also note that $\mathfrak{e}_r$ and $\mathfrak{f}_0$ are, respectively, the ideals generated by the r by r minors of the first r rows and the first r columns of M.

The invariant we define is in terms of mixed multiplicities, so we next recall some facts on mixed multiplicities of sets of ideals. These were introduced for two ideals by Bhattacharya [1] and later also by Teissier [8], and more recently the definition was extended to a set of d ideals, where d is the dimension of the ring by Rees (see [3]). We briefly recall the situation we need for our construction. This appears to by slightly different than that considered by Rees; he considered d ideals (not necessarily distinct) such that it is possible to choose one element from each of the ideals to form a system of parameters for the ring A. We require instead that at least one of the ideals be $\mathfrak{m}$–primary. So let $\mathfrak{a}_1,...,\mathfrak{a}_n$ be n ideals of A such that $\mathfrak{a}_l$ is $\mathfrak{m}$–primary. If all of the ideals were $\mathfrak{m}$–primary, there would be a polynomial P in n variables of degree d such that we would have

$$P(s_1,...,s_n) = \text{length}(A/\mathfrak{a}_1^{s_1}\mathfrak{a}_2^{s_2}...\mathfrak{a}_n^{s_n})$$

for large values of $s_1,...,s_n$. In our case these lengths are not finite, so this does not make sense. However, since $\mathfrak{a}_l$ is $\mathfrak{m}$–primary, there is still a polynomial P' in n variables of degree $d-1$ such that we have

$$P'(s_1,...,s_n) = \text{length}(\mathfrak{a}_1^{s_1}\mathfrak{a}_2^{s_2}...\mathfrak{a}_n^{s_n}/\mathfrak{a}_1^{s_1+1}\mathfrak{a}_2^{s_2}...\mathfrak{a}_2^{s_2}...\mathfrak{a}_n^{s_n})$$

for large values of $s_1,...,s_n$. In the case in which all ideals are $\mathfrak{m}$–primary, this is the difference $P(s_1 + 1,...,s_n) - P(s_1,...,s_n)$ and one can recover those coefficients of P which invove at least one factor of $\mathfrak{a}_1$. In our case, this gives a well–defined coefficient for each term of the polynomial for which at least one $\mathfrak{m}$–primary factor occurs. We summarize this in the following definition :

DEFINITION. *Let* $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$ *be* n *ideals of* A *such that* $\mathfrak{a}_1,\ldots,\mathfrak{a}_k$ *are* $\mathfrak{m}$*–primary. We call the mixed multiplicity polynomial of* $\mathfrak{a}_1,\ldots,\mathfrak{a}_k;\ \mathfrak{a}_{k+1},\ldots,\mathfrak{a}_n$ *the homogeneous polynomial* P *in* n *variables of degree* d *such that*

(1) *for* $i = 1,\ldots,k$ *we have* $P(s_1,\ldots,s_i+1,\ldots,s_n) - P(s_1,\ldots,s_i,\ldots,s_n) =$ *the part of degree* $d-1$ *of the polynomial which gives the length of* $\mathfrak{a}_1^{s_1}\ldots\mathfrak{a}_i^{s_i}\ldots\mathfrak{a}_n^{s_n}/\mathfrak{a}_1^{s_1}\ldots\mathfrak{a}_i^{s_i+1}\ldots\mathfrak{a}_n^{s_n}$. *For large* $s_1,\ldots,s_n$,

(2) *all coefficients involving only the last* $n-k$ *variables are zero.*

We make two remarks on this definition. First, it might seem reasonable to call it the Hilbert–Samuel polynomial in analogy with the case of one ideal; the terminology we have chosen is because we have taken only the part of degree d, and these coefficients are (up to certain multinomial coefficients) the mixed multiplicities of the ideals. The second is that the last condition, letting those coefficients which are not well–defined be zero, may seem arbitrary, but it turns out to be exactly what is needed in our formula.

We give an alternative description of the coefficients of the polynomial which will be useful later. We begin by taking the multigraded ring whose $s_1,\ldots,s_n$ component is $\mathfrak{a}_1^{s_1}\mathfrak{a}_2^{s_2}\ldots\mathfrak{a}_n^{s_n}$. In conformity with the usual terminology for one ideal, we call this the Rees ring associated to $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$. By taking the projective scheme associated to this, one gets a scheme X proper over Spec A with an imbedding into the product of projective space over Spec(A); this imbedding is defined by choosing a set of generators for each of the ideals. Finally, on X there are invertible sheaves of ideals $O(-A_1),\ldots,O(-A_n)$ associated to divisors $A_1,\ldots,A_n$ defined by the ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$. The coefficients of the mixed multiplicity polynomial can then be defined as the degrees of the intersections of these divisors. More precisely, one has coefficient of

$$s_1^{k_1}s_2^{k_2}\ldots s_n^{k_n} = (-1)^{d-1}\Big(\frac{1}{k_1!\ldots k_n!}\Big)A_1^{k_1}A_2^{k_2},\ldots,A_n^{k_n}.$$

In this intersection product one must first take the exceptional divisor corresponding to an ideal which is $\mathfrak{m}$–primary, which reduces the situation to a subscheme which lies over the closed point of Spec(A), and then intersect with the other divisors. In ring–theoretic terms, this can be done by first dividing the Rees ring by the image of one of the ideals which is $\mathfrak{m}$–primary, which reduces the situation to a multigraded ring over an Artinian ring, and then dividing by generic enough elements in appropriate graded pieces of the Rees ring (this works at least if the residue field of A is infinite). The sign occurs because every intersection after the first is with one of the hyperplanes coming from the embedding into a product of projective spaces, and this is the negative of the corresponding exceptional divisor. The mixed multiplicity polynomial can thus be expressed more simply as