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# THE HOMOLOGICAL THEORY OF MAXIMAL COHEN-MACAULAY APPROXIMATIONS

by

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*Summary*: Let R be a commutative noetherian Cohen-Macaulay ring which admits a dualizing module. We show that for any finitely generated R-module N there exists a maximal Cohen-Macaulay R-module M which surjects onto N and such that any other surjection from a maximal Cohen-Macaulay module onto N factors over it. Dually, there is a finitely generated R-module I of finite injective dimension into which N embeds, universal for such embeddings. We prove and investigate these results in the broader context of abelian categories with a suitable subcategory of "maximal Cohen-Macaulay objects" extracting for this purpose those ingredients of Grothendieck-Serre duality theory which are needed.

*Résumé*: Soit R un anneau commutatif, noethérien et de Cohen-Macaulay, tel que un module dualisant existe pour R. On démontre que pour chaque R-module N de type fini il existe un R-module M de profondeur maximale et un homomorphisme surjectif de M sur N, tel que toute autre surjection d'un tel module sur N s'en factorise. De manière duale, il existe aussi un plongement de N dans un R-module I de type fini et de dimension injective finie, universelle pour telles plongements. Nous démontrons et examinons ces résultats dans le cadre des catégories abéliennes avec une sous-catégorie convenable des "objets de Cohen-Macaulay maximaux", à cet effet mettant en évidence les propriétés de la théorie de dualité de Grothendieck-Serre dont on a besoin.

## §0. *A Commutative Introduction*

The aim of this work is to analyze the framework in which the theory of *maximal Cohen-Macaulay approximations* can be developed. Instead of outlining right away the abstract results, we want to start by describing the situation in the classical case of a commutative local noetherian ring R with maximal ideal $\mathbf{m}$ and residue class field $k = R/\mathbf{m}$.

Assume that R admits a *dualizing module* $\omega$. Then R is Cohen-Macaulay, and the finitely generated R-modules M which are maximal Cohen-Macaulay in the sense that $\operatorname{depth}_{\mathbf{m}} M = \dim R$ can be characterized homologically as those modules for which $\operatorname{Ext}^i_R(M, \omega) = 0$ for $i \neq 0$.

Our main result can then be paraphrased as saying that **R-mod**, the category of finitely generated R-modules, is obtained by glueing together the orthogonal subcategories of *modules of finite injective dimension* over R and the category of *maximal Cohen-Macaulay modules* along their common intersection which is spanned by $\omega$.

More precisely, let us recall that $\omega$ is *dualizing* for the local ring $(R,\mathfrak{m},k)$ if and only if it satisfies the following three conditions:

(i) $\omega$ is finitely generated and of finite injective dimension over R.

(ii) The natural ring homomorphism which is given by multiplication with scalars from R on $\omega$, $R \to \mathrm{Hom}_R(\omega,\omega)$ is an isomorphism.

(iii) For any integer $i \neq 0$, one has $\mathrm{Ext}^i_R(\omega,\omega) = 0$.

Now our main results in this context are

<u>Theorem A:</u> *(Existence of the decomposition).* Let $(R,\mathfrak{m},k)$ be a commutative, local noetherian ring with dualizing module $\omega$. For any finitely generated R-module N there exist finitely generated R-modules $M_N$ and $I^N$ together with an R-linear map

$$d_N : M_N \to I^N$$

such that

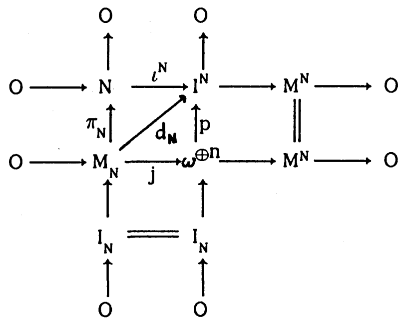
(a) The image of $d_N$ is isomorphic to N.

(b) $M_N$ is maximal Cohen-Macaulay and $I_N = \mathrm{Ker}\, d_N$ is an R-module of finite injective dimension.

(c) $I^N$ is of finite injective dimension and $M^N = \mathrm{Cok}\, d_N$ is maximal Cohen-Macaulay.

(d) There exists an integer $n \geq 0$ such that $d_N$ can be factored into an injection $j : M_N \to \omega^{\oplus n}$ and a surjection $p : \omega^{\oplus n} \to I^N$. ■

If $d_N = \iota^N \circ \pi_N$ denotes the factorization of $d_N$ over its image N, we can arrange the data given in the theorem into the following exact commutative diagram of R-modules:



*Theorem B: (Essential Uniqueness)*

(a) Assume given a second homomorphism $d'_N: M'_N \to {}'I^N$ satisfying Theorem A for the same module N. If the image factorization of $d'_N$ is given as

$$M'_N \xrightarrow{\pi'_N} N \xrightarrow{\iota'^N} {}'I^N,$$

there exist modules P, P' and Q, 'Q which are each finite direct sums of copies of $\omega$, and R-module isomorphisms $\mu$, $\kappa$ so that the following diagram commutes:

$$\begin{array}{ccccc} M'_N\oplus P & \xrightarrow{\pi'_N\oplus 0} & N & \xrightarrow{\iota'^N\oplus 0} & {}'I^N\oplus Q \\ \mu\downarrow & & \| & & \downarrow\kappa \\ M_N\oplus P' & \xrightarrow{\pi_N\oplus 0} & N & \xrightarrow{\iota^N\oplus 0} & I^N\oplus {}'Q \end{array}$$

(b) If $f:M \to N$ is any homomorphism from a maximal Cohen-Macaulay R-module M into N, it factors over $\pi_N$. If $g:N \to J$ is any homomorphism from N into an R-module J of finite injective dimension, it factors over $\iota^N$. ∎

These results suggest to call $0 \to I_N \to M_N \xrightarrow{\pi_N} N \to 0$ a *maximal Cohen-Macaulay approximation of* N and $0 \to N \xrightarrow{\iota^N} I^N \to M^N \to 0$ a *hull of finite injective dimension* for N.

To give a simple illustration, consider the case where N itself is a Cohen-Macaulay R-module, hence satisfying $\text{depth}_{\mathfrak{m}} N = \dim N$.
Set $n = \text{codepth}_{\mathfrak{m}} N = \dim R - \dim N$. Then local duality theory implies:

(i) $\text{Ext}^i_R(N,\omega) = 0$ for $i \neq n$.

(ii) $N^\vee = \text{Ext}^n_R(N,\omega)$ is again Cohen-Macaulay of codepth n.

(iii) $N = \text{Ext}^n_R(N^\vee,\omega) = N^{\vee\vee}$.

Using this information, let

$$0 \to \Omega_n(N) \to R^{\oplus b_{n-1}} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_0} R^{\oplus b_0} \longrightarrow N^\vee \to 0$$

be an exact sequence obtained by truncating a free resolution of $N^\vee$. It follows that $\Omega_n(N)$ is maximal Cohen-Macaulay and that dualizing with respect to $\omega$ results in an exact sequence

$$0 \longrightarrow \omega^{\oplus b_0} \longrightarrow \dots \xrightarrow{d^\vee_{n-2}} \omega^{\oplus b_{n-1}} \longrightarrow \text{Hom}_R(\Omega_n(N),\omega) \xrightarrow{\pi} N^{\vee\vee} = N \longrightarrow 0.$$

Then $M_n = \text{Hom}_R(\Omega_n(N),\omega) \xrightarrow{\pi} N$ is a desired maximal Cohen-Macaulay approximation of N, and $I_N = \text{Cok}\ d^\vee_{n-2}$ admits a finite resolution "by $\omega$", which shows that $I_N$ is of finite injective dimension. The hull of finite injective dimension $I^N$ is then simply the cokernel of the $\omega$-dual of the next differential in the resolution of $N^\vee$, namely
$I^N = \text{Cok Hom}_R(d_{n-1},\omega)$.

If R is a domain, for example, we get even more precise information:

(i) The rank of $M_N$ equals the alternating sum

$$\sum_{i=1}^{n}(-1)^{i+1}b_{n-i} + (-1)^n \text{ rk } N,$$

(ii) $M_N^{\vee} = \text{Hom}_R(M_N, \omega) = \Omega_n(N)$ embeds into $R^{\oplus b_{n-1}}$,

(iii) $M_N$ contains no copy of $\omega$ as a direct summand if and only if $\Omega_n(N)$, the n-th syzygy module of N, contains no free summand.

It follows that one can attach new numerical invariants to an R-module N in this way. The minimum number of copies of $\omega$ necessarily contained in $M_N$ or $I^N$, the rank of the $\omega$-free summand of either $M_N$ or $I^N$, their minimum number of generators and so forth.

Here, we are not concerned with these more detailed consequences of the theory but rather with its general framework.

The first author first proved an essentially equivalent version of Theorem A but for the category of additive functors on **R-mod**, see [Ausl], where the result was phrased by saying that the category of maximal Cohen-Macaulay modules is "coherently (co-)finite". The essential step then was to establish the representability of the functors involved.

This background illuminates our approach here. Although the primary applications of the theory might be within the classical theory of rings and algebras, to a large extent it can be developed in any abelian category **C** which admits a suitable subcategory **X** of "maximal Cohen-Macaulay objects".

Here we establish sufficient conditions on **X** to guarantee the categorical analogues of Theorems A and B. Section 1 deals with the decomposition theorem and section 2 addresses the uniqueness question. Sections 3 and 4 investigate the circumstances under which - in the terminology of the above example - the category of modules with "finite $\omega$-resolution" are *all* the modules of finite injective dimension. Section 5 assembles a few remarks on finiteness conditions and section 6 contains more examples, among other purposes highlighting the differences in the theory when applied to either commutative or non-commutative rings.

## §1. *The Basic Decomposition Theorem*

In this section we prove the basic decomposition theorem on which this paper rests. Before stating the result, we give some definitions and notations.

Throughout, **C** will be an *abelian* category. By a *subcategory* **A** of **C** we will always mean a *full, additive* and *essential* subcategory of **C**, so that **A** is closed under finite direct sums in **C** and such that any object C in **C** which is isomorphic to an object in **A** is already an object in **A**.

A subcategory of **C** is said to be *additively closed* (or *karoubian* in the