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WITTEN LAPLACIAN ON A LATTICE SPIN SYSTEM

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60^th birthday

Abstract. — We consider an unbounded lattice spin system with a Gibbs measure. We introduce the Hodge-Kodaira operator acting on differential forms and give a sufficient condition for the positivity of the lowest eigenvalue.

***Résumé* (Laplacien de Witten sur un système de spin sur réseau).** — Nous considérons un réseau de spin muni d'une mesure de Gibbs. Nous introduisons l'opérateur de Hodge-Kodaira agissant sur les formes différentielles, et nous donnons une condition suffisante pour la positivité de la plus petite valeur propre.

1. Introduction

In this paper, we consider the spectral gap problem for a lattice spin system. Here, in our case, the single spin space is $\mathbb{R}$ and so it is non-compact. This is sometimes called an unbounded spin system.

We consider a model that each spin sits on the lattice $\mathbb{Z}^d$, and so the configuration space is $\mathbb{R}^{\mathbb{Z}^d}$. We suppose that a Gibbs measure is given in $\mathbb{R}^{\mathbb{Z}^d}$, which has the following formal expression:

$$\nu = Z^{-1} \exp\Big\{-2\mathscr{J} \sum_{\substack{i,j\in\mathbb{Z}^d \\ i\sim j}} (x^i - x^j)^2 - 2\sum_{i\in\mathbb{Z}^d} U(x^i)\Big\} \prod_{i\in\mathbb{Z}^d} dx^i. \tag{1.1}$$

Here U is a function of $\mathbb{R}$, called a self potential and $i \sim j$ means that $\|i-j\|_1 = \sum_k |i_k - j_k| = 1$. Under this measure we define the Hodge-Kodaira operator and discuss the positivity of the lowest eigenvalue of the operator. For unbounded spin

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systems, the Poincaré inequality, the logarithmic Sobolev inequality and other properties are well discussed, e.g., Zegarlinski [**11**], Yoshida [**10**], etc. In particular, Helffer [**5, 6, 7, 8**] dealt with this problem in connection to the Witten Laplacian. In fact, he proved the positivity of the lowest eigenvalue of the Hodge-Kodaira operator acting on 1-forms. From this point of view, we generalize his result to any p-forms ($p \geq 1$), i.e., we will prove that the lowest eigenvalue of the Hodge-Kodaira operator acting on p-forms is positive.

The organization of the paper is as follows. In Section 2, we discuss the Witten Laplacian on a finite dimensional space and in Section 3, we summarize differential forms, the Hodge-Kodaira operator and the Weitzenböck formula, which is crucial in the later argument. In Section 4, we give an estimate of spectral gap for 1-dimensional case. Last in Section 5, we prove the positivity of the lowest eigenvalue of the Hodge-Kodaira operator. We only consider the finite region case but we give a uniform estimate. In fact, it is independent of the choice of region and the boundary condition. So the result is valid for the infinite volume case as well.

2. Witten Laplacian in finite dimension

We give a quick review of the Witten Laplacian, which we need later. Details and related topics can be found in Hellfer [**8**], Albeverio-Daletskii-Kondratiev [**1**], Elworthy-Rosenberg [**4**], etc. Simon et al [**3**] is also a good reference for the supersymmetry.

Our interest is in the infinite dimensional case, but we start with the finite dimensional case. Suppose we are given a C^2 function Φ on $\mathbb{R}^N$ and define a measure ν by

$$\nu(dx) = Z^{-1} e^{-2\Phi} dx. \tag{2.1}$$

Here $Z = \int_{\mathbb{R}^N} e^{-2\Phi} dx$ so that ν is a probability measure. Define a Dirichlet form $\mathscr{E}$ by

$$\mathscr{E}(f,g) = \int_{\mathbb{R}^N} (\nabla f, \nabla g) e^{-2\Phi} dx, \tag{2.2}$$

where $\nabla = (\partial_1, \dots, \partial_N)$, $\partial_k = \frac{d}{dx_k}$. $(\nabla f, \nabla g)$ stands for the Euclidean inner product. We must specify the domain of $\mathscr{E}$. (2.2) is well-defined for f, $g \in C_0^\infty(\mathbb{R}^N)$. So at first, $\mathscr{E}$ is defined on $C_0^\infty(\mathbb{R}^N)$. Let us give an explicit form of the dual operator ∂_j^* of ∂_j in $L^2(\nu)$. To do this, note that

$$\int_{\mathbb{R}^N} \partial_j f g e^{-2\Phi} dx = -\int_{\mathbb{R}^N} f \partial_j (g e^{-2\Phi}) dx = -\int_{\mathbb{R}^N} f(\partial_j g - 2\partial_j \Phi g) e^{-2\Phi} dx,$$

which means

$$\partial_j^* = -\partial_j + 2\partial_j \Phi. \tag{2.3}$$

Here ∂_j^* is the dual operator of ∂_j in $L^2(\nu)$.

From this, we can see that the dual operator of ∇ has dense domain and so ∇ is closable. Moreover the generator $\mathfrak{A}$ is given by

$$\mathfrak{A}f = -\sum_j \partial_j^*\partial_j = \sum_j (\partial_j^2 f - 2\partial_j\Phi\partial_j f) = \triangle f - 2(\nabla\Phi, \nabla f). \tag{2.4}$$

This is valid for $f \in C_0^\infty(\mathbb{R}^N)$. We can show that $\mathfrak{A}$ is essentially self-adjoint and so, by taking closure, we may regard $\mathfrak{A}$ as self-adjoint operator. The domain of $\mathscr{E}$ is a set of all functions $f \in L^2(\nu)$ with $\nabla f \in L^2(\nu; \mathbb{R}^N)$.

We now define a Witten Laplacian. Let $I\colon L^2(dx) \longrightarrow L^2(\nu)$ be a unitary operator defined by

$$If(x) = e^\Phi f. \tag{2.5}$$

Let us obtain a operator X_j which satisfies the following commutative diagram:

$$\begin{tikzcd} L^2(dx) \arrow[r, "I"] \arrow[d, "X_j"'] & L^2(\nu) \arrow[d, "\partial_j"] \\ L^2(dx) \arrow[r, "I"] & L^2(\nu) \end{tikzcd} \tag{2.6}$$

It is not hard to see that

$$X_j = e^{-\Phi}\partial_j e^\Phi = \partial_j + \partial_j\Phi.$$

We denote the dual operator of X_j in $L^2(dx)$ by $\tilde{X}_j$. Here we use the following convention. $*$ stands for the dual operator in $L^2(\nu)$ and $\tilde{}$ stands for the dual operator in $L^2(dx)$, dx being the Lebesgue measure in $\mathbb{R}^N$. $\tilde{X}_j$ has the following form:

$$\tilde{X}_j = -\partial_j + \partial_j\Phi.$$

This is also equal to $e^{-\Phi}\partial_j^* e^\Phi$. The operator A associated with the generator $\mathfrak{A} = -\sum_j \partial_j^*\partial_j$ is computed by

$$\begin{aligned} A = e^{-\Phi}\mathfrak{A}e^\Phi &= -e^{-\Phi}\Big(\sum_j \partial_j^*\partial_j\Big)e^\Phi = -\sum_j \tilde{X}_j X_j \\ &= -\sum_j(-\partial_j + \partial_j\Phi)(\partial_j + \partial_j\Phi) = \sum_j(\partial_j^2 + \partial_j^2\Phi - (\partial_j\Phi)^2) \\ &= \triangle + \triangle\Phi - |\nabla\Phi|^2. \end{aligned}$$

Definition 2.1. — $A = \triangle + \triangle\Phi - |\nabla\Phi|^2$ in $L^2(dx)$ is called a Witten Laplacian.

$\mathfrak{A}$ and A are unitarily equivalent to each other but we distinguish them and call A as the Witten Laplacian, which is an operator in $L^2(dx)$.

The following commutation relation is easily checked.

Proposition 2.1. — In $L^2(\nu)$, we have

$$[\partial_i, \partial_j] = 0, \tag{2.7}$$

$$[\partial_i, \partial_j^*] = 2\partial_i\partial_j\Phi, \tag{2.8}$$

$$[\partial_j^*, \partial_k^*] = 0. \tag{2.9}$$

Further, in $L^2(dx)$, we have

$$[X_i, X_j] = 0, \tag{2.10}$$

$$[X_i, \tilde{X}_j] = 2\partial_i\partial_j\Phi, \tag{2.11}$$

$$[\tilde{X}_j, \tilde{X}_j] = 0. \tag{2.12}$$

3. Witten Laplacian acting on differential forms

In Section 2, we have introduced the Witten Laplacian. We now proceed to the Witten Laplacian acting on differential forms.

Let us quickly review the exterior algebra. In the sequel, we will deal with multilinear functionals on $\mathbb{R}^N$. Let t be a p-linear functional and s be a q-linear functional, e.g., t is a functional from $\underbrace{\mathbb{R}^N \times \cdots \times \mathbb{R}^N}_{p}$ into $\mathbb{R}$ which is linear in each coordinate. We define $p+q$-linear functional $t \otimes s$ by

$$t \otimes s(v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}) = t(v_1, \ldots, v_p)s(v_{p+1}, \ldots, v_{p+q}). \tag{3.1}$$

$t \otimes s$ is called a tensor product. We also define the alternation mapping A_p by

$$A_p t(v_1, \ldots, v_p) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (\operatorname{sgn} \sigma)\, t(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \tag{3.2}$$

for p-linear functional t. Here $\mathfrak{S}_p$ is the symmetric group of degree p and $\operatorname{sgn} \sigma$ stands for the signature. If p-linear functional θ satisfies $A_p\theta = \theta$, θ is called alternating. We denote the set of all alternating functionals of degree p by $\bigwedge^p(\mathbb{R}^N)^*$. For $\theta \in \bigwedge^p(\mathbb{R}^N)^*$ and $\eta \in \bigwedge^q(\mathbb{R}^N)^*$, we define their exterior product $\theta \wedge \eta$ by

$$\theta \wedge \eta = \frac{(p+q)!}{p!q!} A_{p+q}(\theta \otimes \eta). \tag{3.3}$$

Taking an orthonormal basis $\theta_1, \ldots, \theta_N$ in $(\mathbb{R}^N)^*$, any element of $\bigwedge^p(\mathbb{R}^N)^*$ is represented as a unique linear combination of the following elements

$$\theta_{i_1} \wedge \cdots \wedge \theta_{i_p}. \tag{3.4}$$

We define an inner product in $\bigwedge^p(\mathbb{R}^N)^*$ so that all elements of the form (3.4) become an orthonormal basis in $\bigwedge^p(\mathbb{R}^N)^*$.

$A^p(\mathbb{R}^N) = \mathbb{R}^N \times \bigwedge^p(\mathbb{R}^N)^*$ has a structure of vector bundle and a section of $A^p(\mathbb{R}^N)$ is called a differential form of degree p. The set of all sections is denoted by $\Gamma(A^p(\mathbb{R}^N))$.

Since the vector bundle $A^p(\mathbb{R}^N)$ is trivial, any section can be identified with a mapping from $\mathbb{R}^N$ into $\bigwedge^p(\mathbb{R}^N)^*$. In the sequel, we use this convention. $\Gamma^\infty(A^p(\mathbb{R}^N))$ denotes the set of all smooth differential forms and $\Gamma_0^\infty(A^p(\mathbb{R}^N))$ denotes the set of all smooth differential forms with compact support.

We introduce some operators in $\bigwedge^p(\mathbb{R}^N)^*$ as follows. For $\theta \in (\mathbb{R}^N)^*$, we define $\mathrm{ext}(\theta)\colon \bigwedge^p(\mathbb{R}^N)^* \longrightarrow \bigwedge^{p+1}(\mathbb{R}^N)^*$ by

$$\mathrm{ext}(\theta)\omega = \theta \wedge \omega \tag{3.5}$$

and for $v \in \mathbb{R}^N$, we define $\mathrm{int}(\theta)\colon \bigwedge^p(\mathbb{R}^N)^* \longrightarrow \bigwedge^{p-1}(\mathbb{R}^N)^*$ by

$$(\mathrm{int}(v)\omega)(v_1, \dots, v_{p-1}) = \omega(v, v_1, \dots, v_{p-1}). \tag{3.6}$$

Taking a standard basis $\{e_1, \dots, e_N\}$ of $\mathbb{R}^N$ and its dual basis $\{\theta^1, \dots, \theta^N\}$, we define operators a^i, $(a^i)^*$ by

$$a^i = \mathrm{int}(e_i), \tag{3.7}$$

$$(a^i)^* = \mathrm{ext}(\theta^i). \tag{3.8}$$

Here we regard a^i, $(a^i)^*$ as operators on an exterior algebra $\mathbb{R} \oplus (\mathbb{R}^N)^* \oplus \bigwedge^2(\mathbb{R}^N)^* \oplus \cdots \oplus \bigwedge^N(\mathbb{R}^N)^*$. They satisfy the following commutation relation:

$$[a^i, a^j]_+ = 0, \tag{3.9}$$

$$[a^i, (a^j)^*]_+ = \delta_{ij}, \tag{3.10}$$

$$[(a^i)^*, (a^j)^*]_+ = 0. \tag{3.11}$$

Here $[\ ,\ ]_+$ stands for an anti-commutator, i.e., $[a^i, a^j]_+ = a^i a^j + a^j a^i$.

For differential forms, the covariant differentiation ∇ can be defined. More generally, the covariant differentiation ∇ is defined for tensor fields as follows:

$$\nabla t = \sum_i \theta^i \otimes \partial_i t.$$

Here we remark that the operator is considered in $L^2(\nu)$, i.e., the reference measure is ν. The dual operator of ∇ in $L^2(\nu)$ is given by

$$\nabla^*(\sum_i \theta^i \otimes t_i) = \sum_i \partial_i^* t_i$$

and so we have

$$\nabla^*\nabla t = \sum_i \partial_i^* \partial_i t = -\sum_i (\partial_i^2 - 2\partial_i \Phi \partial_i) t.$$

For differential forms, we can define the exterior differentiation as follows. Let ω be a differential form of degree p. Then its exterior derivative is defined by $d\omega = (p+1)A_{p+1}\nabla\omega$ and it is written as

$$d = \sum_i \mathrm{ext}(\theta^i)\partial_i = \sum_i (a^i)^* \partial_i. \tag{3.12}$$