Astérisque
327, 2009, p. 131–199

PURE SPINORS ON LIE GROUPS

by

A. Alekseev, H. Bursztyn & E. Meinrenken

Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday.

Abstract. — For any manifold M, the direct sum $\mathbb{T}M = TM \oplus T^*M$ carries a natural inner product given by the pairing of vectors and covectors. Differential forms on M may be viewed as spinors for the corresponding Clifford bundle, and in particular there is a notion of *pure spinor*. In this paper, we study pure spinors and Dirac structures in the case when $M = G$ is a Lie group with a bi-invariant pseudo-Riemannian metric, e.g. G semi-simple. The applications of our theory include the construction of distinguished volume forms on conjugacy classes in G, and a new approach to the theory of quasi-Hamiltonian G-spaces.

***Résumé* (Spineurs purs sur les groupes de Lie).** — Pour toute variété lisse M, le fibré $\mathbb{T}M = TM \oplus T^*M$ est muni d'un produit scalaire naturel défini par la dualité entre vecteurs et co-vecteurs. Les formes différentielles sur M sont des spineurs pour le fibré de Clifford correspondant. On définit alors les *spineurs purs*. Dans cet article, nous étudions les spineurs purs et les structures de Dirac dans le cas où M est un groupe de Lie G muni d'une métrique pseudo-riemannienne bi-invariante, par exemple un groupe semi-simple. Comme applications de notre théorie, nous définissons une forme volume distinguée sur les classes de conjugaison de G, et nous proposons une nouvelle approche de la théorie des G-espaces quasi-hamiltoniens.

0. Introduction

For any manifold M, the direct sum $\mathbb{T}M = TM \oplus T^*M$ carries a non-degenerate symmetric bilinear form, extending the pairing between vectors and covectors. There is a natural Clifford action ϱ of the sections $\Gamma(\mathbb{T}M)$ on the space $\Omega(M) = \Gamma(\wedge T^*M)$ of differential forms, where vector fields act by contraction and 1-forms by exterior multiplication. That is, $\wedge T^*M$ is viewed as a spinor module over the Clifford bundle $\mathrm{Cl}(\mathbb{T}M)$. A form $\phi \in \Omega(M)$ is called a *pure spinor* if the solutions $w \in \Gamma(\mathbb{T}M)$ of

2010 Mathematics Subject Classification. — 53D17, 53D20, 15A66.
Key words and phrases. — Dirac structures, Courant algebroids, Clifford algebras, pure spinors, moment maps.

$\varrho(w)\phi = 0$ span a Lagrangian subbundle $E \subset \mathbb{T}M$. Given a closed 3-form $\eta \in \Omega^3(M)$, a pure spinor ϕ is called *integrable* (relative to η) [**9, 28**] if there exists a section $w \in \Gamma(\mathbb{T}M)$ with

$$(\mathrm{d} + \eta)\phi = \varrho(w)\phi.$$

In this case, there is a generalized foliation of M with tangent distribution the projection of E to TM. The subbundle E defines a *Dirac structure* [**20, 50**] on M, and the triple (M, E, η) is called a *Dirac manifold.*

The present paper is devoted to the study of Dirac structures and pure spinors on Lie groups G. We assume that the Lie algebra $\mathfrak{g}$ carries a non-degenerate invariant symmetric bilinear form B, and take $\eta \in \Omega^3(G)$ as the corresponding Cartan 3-form. Let $\overline{\mathfrak{g}}$ denote the Lie algebra $\mathfrak{g}$ with the opposite bilinear form $-B$. We will describe a trivialization

$$\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \overline{\mathfrak{g}}),$$

under which any Lagrangian Lie subalgebra $\mathfrak{s} \subset \mathfrak{g} \oplus \overline{\mathfrak{g}}$ defines a Dirac structure on G. There is also a similar identification of spinor bundles

$$\mathcal{R}\colon G \times \mathrm{Cl}(\mathfrak{g}) \xrightarrow{\cong} \wedge T^*G,$$

taking the standard Clifford action of $\mathfrak{g} \oplus \overline{\mathfrak{g}}$ on $\mathrm{Cl}(\mathfrak{g})$, where the first summand acts by left (Clifford) multiplication and the second summand by right multiplication, to the Clifford action ϱ. This isomorphism takes the *Clifford differential* d_{Cl} on $\mathrm{Cl}(\mathfrak{g})$, given as Clifford commutator by a cubic element [**4, 38**], to the the differential $\mathrm{d} + \eta$ on $\Omega(G)$. As a result, pure spinors $x \in \mathrm{Cl}(\mathfrak{g})$ for the Clifford action of $\mathrm{Cl}(\mathfrak{g} \oplus \overline{\mathfrak{g}})$ on $\mathrm{Cl}(\mathfrak{g})$ define pure spinors $\phi = \mathcal{R}(x) \in \Omega(G)$, and the integrability condition for ϕ is equivalent to a similar condition for x. The simplest example $x = 1$ defines the *Cartan-Dirac structure* E_G [**14, 50**], introduced by Alekseev, Ševera and Strobl in the 1990's. In this case, the resulting foliation of G is just the foliation by conjugacy classes. We will study this Dirac structure in detail, and examine in particular its behavior under group multiplication and under the exponential map. When G is a complex semi-simple Lie group, it carries another interesting Dirac structure, which we call the *Gauss-Dirac structure.* The corresponding foliation of G has a dense open leaf which is the 'big cell' from the Gauss decomposition of G.

The main application of our study of pure spinors is to the theory of q-Hamiltonian actions [**2, 3**]. The original definition of a q-Hamiltonian G-space in [**3**] involves a G-manifold M together with an invariant 2-form ω and a G-equivariant map $\Phi\colon M \to G$ satisfying appropriate axioms. As observed in [**14, 15**], this definition is equivalent to saying that the 'G-valued moment map' Φ is a suitable morphism of Dirac manifolds (in analogy with classical moment maps, which are morphisms $M \to \mathfrak{g}^*$ of Poisson manifolds). In this paper, we will carry this observation further, and develop all the basic results of q-Hamiltonian geometry from this perspective. A conceptual advantage of this alternate viewpoint is that, while the arguments in [**3**] required G to be compact, the Dirac geometry approach needs no such assumption, and in fact works in the complex (holomorphic) category as well. This is relevant for applications: For instance, the symplectic form on a representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ (for Σ a closed surface) can be obtained by q-Hamiltonian reduction, and there are many interesting examples for noncompact G. (For instance, the case $G = \mathrm{PSL}(2,\mathbb{R})$ gives the symplectic form on Teichmüller space.) Complex q-Hamiltonian spaces appear e.g. in the work of Boalch [**13**] and Van den Bergh [**11**].

The organization of the paper is as follows. Sections 1 and 2 contain a review of Dirac geometry, first on vector spaces and then on manifolds. The main new results in these sections concern the geometry of Lagrangian splittings $\mathbb{T}M = E \oplus F$ of the bundle $\mathbb{T}M$. If $\phi, \psi \in \Omega(M)$ are pure spinors defining E, F, then, as shown in [**17, 19**], the top degree part of $\phi^\top \wedge \psi$ (where $\top$ denotes the standard anti-involution of the exterior algebra) is nonvanishing, and hence defines a volume form μ on M. Furthermore, there is a bivector field $\pi \in \mathfrak{X}^2(M)$ naturally associated with the splitting, which satisfies

$$\phi^\top \wedge \psi = e^{-\iota(\pi)}\mu.$$

We will discuss the properties of μ and π in detail, including their behavior under Dirac morphisms.

In Section 3 we specialize to the case $M = G$, where G carries a bi-invariant pseudo-Riemannian metric, and our main results concern the isomorphism $\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \bar{\mathfrak{g}})$ and its properties. Under this identification, the Cartan-Dirac structure $E_G \subset \mathbb{T}G$ corresponds to the diagonal $\mathfrak{g}_\Delta \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$, and hence it has a natural Lagrangian complement $F_G \subset \mathbb{T}G$ defined by the anti-diagonal. We will show that the exponential map gives rise to a Dirac morphism $(\mathfrak{g}, E_\mathfrak{g}, 0) \to (G, E_G, \eta)$ (where $E_\mathfrak{g}$ is the graph of the linear Poisson structure on $\mathfrak{g} \cong \mathfrak{g}^*$), but this morphism does not relate the obvious complements $F_\mathfrak{g} = T\mathfrak{g}$ and F_G. The discrepancy is given by a 'twist', which is a solution of the *classical dynamical Yang-Baxter equation.* For G complex semi-simple, we will construct another Lagrangian complement of E_G, denoted by $\widehat{F}_G$, which (unlike F_G) is itself a Dirac structure. The bivector field corresponding to the splitting $E_G \oplus \widehat{F}_G$ is then a Poisson structure on G, which appeared earlier in the work of Semenov-Tian-Shansky [**49**].

In Section 4, we construct an isomorphism $\wedge T^*G \cong G \times \mathrm{Cl}(\mathfrak{g})$ of spinor modules, valid under a mild topological assumption on G (which is automatic if G is simply connected). This allows us to represent the Lagrangian subbundles E_G, F_G and $\widehat{F}_G$ by explicit pure spinors ϕ_G, ψ_G, and $\widehat{\psi}_G$, and to derive the differential equations controlling their integrability. We show in particular that the Cartan-Dirac spinor satisfies

$$(\mathrm{d} + \eta)\phi_G = 0.$$

Section 5 investigates the foundational properties of q-Hamiltonian G-spaces from the Dirac geometry perspective. Our results on the Cartan-Dirac structure give a direct construction of the fusion product of q-Hamiltonian spaces. On the other hand, we use the bilinear pairing of spinors to show that, for a q-Hamiltonian space (M, ω, Φ), the top degree part of $e^\omega \Phi^* \psi_G \in \Omega(M)$ defines a volume form μ_M. This volume form was discussed in [**8**] when G is compact, but the discussion here applies equally well to non-compact or complex Lie groups. Since conjugacy classes in G are examples of q-Hamiltonian G-spaces, we conclude that for any simply connected Lie group G with bi-invariant pseudo-Riemannian metric (e.g. G semi-simple), *any conjugacy class in G carries a distinguished invariant volume form.* If G is complex semi-simple, one obtains the same volume form μ_M if one replaces ψ_G with the Gauss-Dirac spinor $\widehat{\psi}_G$. However, the form $e^{\omega}\Phi^*\widehat{\psi}_G$ satisfies a nicer differential equation, which allows us to compute the volume of M, and more generally the measure $\Phi_*|\mu_M|$, by Berline-Vergne localization [**12**]. We also explain in this Section how to view the more general q-Hamiltonian q-Poisson spaces [**2**] in our framework.

Lastly, in Section 6, we revisit the theory of K^*-valued moment maps in the sense of Lu [**42**] and its connections with P-valued moment maps [**3**, Sec. 10] from the Dirac geometric standpoint.

Acknowledgments. — We would like to thank Marco Gualtieri and Shlomo Sternberg for helpful comments on an earlier version of this paper. The research of A.A. was supported in part by the Swiss National Science Foundation, while E.M. was supported by an NSERC Discovery Grant and a Steacie Fellowship. H.B. acknowledges financial support from CNPq, and thanks University of Toronto, the Fields Institute and Utrecht University for hospitality at various stages of this project. We also thank the Erwin Schrödinger Institut and the Forschungsinstitut Oberwolfach where part of this work was carried out.

Notation. — Our conventions for Lie group actions are as follows: Let G be a Lie group (not necessarily connected), and $\mathfrak{g}$ its Lie algebra. A G-action on a manifold M is a group homomorphism $\mathscr{A}\colon G \to \mathrm{Diff}(M)$ for which the action map $G\times M \to M,\ (g,m)\mapsto \mathscr{A}(g)(m)$ is smooth. Similarly, a $\mathfrak{g}$-action on M is a Lie algebra homomorphism $\mathscr{A}\colon \mathfrak{g} \to \mathfrak{X}(M)$ for which the map $\mathfrak{g}\times M \to TM,\ (\xi,m)\mapsto \mathscr{A}(\xi)_m$ is smooth. Given a G-action $\mathscr{A}$, one obtains a $\mathfrak{g}$-action by the formula $\mathscr{A}(\xi)(f) = \frac{\partial}{\partial t}\big|_{t=0}\mathscr{A}(\exp(-t\xi))^*f$, for $f\in C^\infty(M)$ (here vector fields are viewed as derivations of the algebra of smooth functions).

1. Linear Dirac geometry

The theory of Dirac manifolds was initiated by Courant and Weinstein in [**20, 21**]. We briefly review this theory, developing and expanding the approach via pure spinors advocated by Gualtieri [**28**] (see also Hitchin [**32**] and Alekseev-Xu [**9**]). All vector spaces in this section are over the ground field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. We begin with some background material on Clifford algebras and spinors (see e.g. [**19**] or [**47**].)

1.1. Clifford algebras. — Suppose V is a vector space with a non-degenerate symmetric bilinear form B. We will sometimes refer to such a bilinear form B as an *inner product* on V. The *Clifford algebra* over V is the associative unital algebra generated by the elements of V, with relations

$$vv' + v'v = B(v, v')\,1.$$

It carries a compatible $\mathbb{Z}_2$-grading and $\mathbb{Z}$-filtration, such that the generators $v \in V$ are odd and have filtration degree 1. We will denote by $x \mapsto x^\top$ the canonical anti-automorphism of exterior and Clifford algebras, equal to the identity on V. For any $x \in \mathrm{Cl}(V)$, we denote by $l^{\mathrm{Cl}}(x), r^{\mathrm{Cl}}(x)$ the operators of graded left and right multiplication on $\mathrm{Cl}(V)$:

$$l^{\mathrm{Cl}}(x)x' = xx', \quad r^{\mathrm{Cl}}(x)x' = (-1)^{|x||x'|}x'x.$$

Thus $l^{\mathrm{Cl}}(x) - r^{\mathrm{Cl}}(x)$ is the operator of graded commutator $[x, \cdot]_{\mathrm{Cl}}$.

The *quantization map* $q\colon \wedge V \to \mathrm{Cl}(V)$ is the isomorphism of vector spaces defined by $q(v_1 \wedge \cdots \wedge v_r) = v_1 \cdots v_r$ for pairwise orthogonal elements $v_i \in V$. Let

$$\mathrm{str}\colon \mathrm{Cl}(V) \to \det(V) := \wedge^{\mathrm{top}}(V)$$

be the *super-trace*, given by q^{-1}, followed by taking the top degree part. It has the property $\mathrm{str}([x, x']_{\mathrm{Cl}}) = 0$.

A *Clifford module* is a vector space S together with an algebra homomorphism $\varrho\colon \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$. If S is a Clifford module, one has a dual Clifford module given by the dual space S^* with Clifford action $\varrho^*(x) = \varrho(x^\top)^*$.

Recall that $\mathrm{Pin}(V)$ is the subgroup of $\mathrm{Cl}(V)^\times$ generated by all $v \in V$ whose square in the Clifford algebra is $vv = \pm 1$. It is a double cover of the orthogonal group $\mathrm{O}(V)$, where $g \in \mathrm{Pin}(V)$ takes $v \in V$ to $(-1)^{|g|}gvg^{-1}$, using Clifford multiplication. The *norm homomorphism* for the Pin group is the group homomorphism

$$\mathsf{N}\colon \mathrm{Pin}(V) \to \{-1, +1\}, \quad \mathsf{N}(g) = g^\top g = \pm 1. \tag{1}$$

Let $\{\cdot,\cdot\}$ be the graded Poisson bracket on $\wedge V$, given on generators by $\{v_1, v_2\} = B(v_1, v_2)$. Then $\wedge^2 V$ is a Lie algebra under the Poisson bracket, isomorphic to $\mathfrak{o}(V)$ in such a way that $\varepsilon \in \wedge^2 V$ corresponds to the linear map $v \mapsto \{\varepsilon, v\}$. The Lie algebra $\mathfrak{pin}(V) \cong \mathfrak{o}(V)$ is realized as the Lie subalgebra $q(\wedge^2(V)) \subset \mathrm{Cl}(V)$.

A subspace $E \subset V$ is called *isotropic* if $E \subset E^\perp$ and *Lagrangian* if $E = E^\perp$. The set of Lagrangian subspaces is non-empty if and only if the bilinear form is *split*. If $\mathbb{K} = \mathbb{C}$, this just means that $\dim V$ is even, while for $\mathbb{K} = \mathbb{R}$ this requires that the bilinear form has signature (n, n). From now on, we will reserve the letter W for a vector space with split bilinear form $\langle\cdot,\cdot\rangle$. We denote by $\mathrm{Lag}(W)$ the Grassmann manifold of Lagrangian subspaces of W. It carries a transitive action of the orthogonal group $\mathrm{O}(W)$.

Remark 1.1. — Suppose $\mathbb{K} = \mathbb{R}$, and identify $W \cong \mathbb{R}^{2n}$ with the standard bilinear form of signature (n, n). The group $\mathrm{O}(W) \cong \mathrm{O}(n, n)$ has maximal compact subgroup $\mathrm{O}(n) \times \mathrm{O}(n)$. Already the subgroup $\mathrm{O}(n) \times \{1\}$ acts transitively on $\mathrm{Lag}(W)$, and in fact the action is free. It follows that $\mathrm{Lag}(W)$ is diffeomorphic to $\mathrm{O}(n)$. Further details may be found in [**46**].