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# INDEX, ETA AND RHO INVARIANTS ON FOLIATED BUNDLES

*by*

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*Dedicated to Jean-Michel Bismut on the occasion of his sixtieth birthday*

***Abstract.*** — We study primary and secondary invariants of leafwise Dirac operators on foliated bundles. Given such an operator, we begin by considering the associated regular self-adjoint operator $\mathcal{D}_m$ on the maximal Connes-Skandalis Hilbert module and explain how the functional calculus of $\mathcal{D}_m$ encodes both the leafwise calculus and the monodromy calculus in the corresponding von Neumann algebras. When the foliation is endowed with a holonomy invariant transverse measure, we explain the compatibility of various traces and determinants. We extend Atiyah's index theorem on Galois coverings to foliations. We define a foliated rho-invariant and investigate its stability properties for the signature operator. Finally, we establish the foliated homotopy invariance of such a signature rho-invariant under a Baum-Connes assumption, thus extending to the foliated context results proved by Neumann, Mathai, Weinberger and Keswani on Galois coverings.

***Résumé* (Indices, invariants êta et rho de fibrés feuilletés).** — Nous étudions certains invariants primaires et secondaires associés aux opérateurs de Dirac le long des feuilles de fibrés feuilletés. Etant donné un tel opérateur, nous considérons d'abord l'opérateur auto-adjoint régulier $\mathcal{D}_m$ qui lui est associé sur le module de Hilbert maximal de Connes-Skandalis, puis nous expliquons comment le calcul fonctionnel de $\mathcal{D}_m$ permet de coder le calcul longitudinal ainsi que le calcul sur les fibres de monodromie dans les algèbres de von Neumann correspondantes. Lorsque le feuilletage admet une mesure transverse invariante par holonomie, nous expliquons la compatibilité de diverses traces et déterminants. Nous étendons le théorème de l'indice pour les revêtements d'Atiyah aux feuilletages. Nous définissons l'invariant rho feuilleté et étudions ses propriétés de stabilité lorsque l'opérateur en question est l'opérateur de signature. Finalement, nous établissons l'invariance par homotopie feuilletée de l'invariant rho de l'opérateur de signature le long des feuilles sous une hypothèse de Baum-Connes, prolongeant ainsi au contexte feuilleté des résultats prouvés par Neumann, Mathai, Weinberger et Keswani dans le cadre des revêtements galoisiens.

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## Introduction and main results

The Atiyah-Singer index theorem on closed compact manifolds is regarded nowadays as a classic result in mathematics. The original result has branched into several directions, producing new ideas and new results. One of these directions consists in considering elliptic differential operators on the following hierarchy of geometric structures:

- fibrations and operators that are elliptic in the fiber directions; for example, a product fibration $M \times T \to T$ and a family $(D_\theta)_{\theta \in T}$ of elliptic operators on $M$ continuously parametrized by $T$;
- Galois $\Gamma$-coverings and $\Gamma$-equivariant elliptic operators;
- measured foliations and operators that are elliptic along the leaves;
- general foliations and, again, operators that are elliptic along the leaves.

One pivotal example, going through all these situations, is the one of foliated bundles. Let $\Gamma \to \tilde{M} \to M$ be a Galois $\Gamma$-cover of a smooth compact manifold $M$, let $T$ be a compact manifold on which $\Gamma$ acts by diffeomorphism. We can consider the diagonal action of $\Gamma$ on $\tilde{M} \times T$ and the quotient space $V := \tilde{M} \times_\Gamma T$, which is a compact manifold, a bundle over $M$ and carries a foliation $\mathcal{F}$. This foliation is obtained by considering the images of the fibers of the trivial fibration $\tilde{M} \times T \to T$ under the quotient map $\tilde{M} \times T \to \tilde{M} \times_\Gamma T$ and is known as a *foliated bundle*. More generally, we could allow $T$ to be a compact topological space with an action of $\Gamma$ by homeomorphisms, obtaining what is usually called a *foliated space* or a *lamination*. We then consider a family of elliptic differential operators $(\tilde{D}_\theta)_{\theta \in T}$ on the product fibration $\tilde{M} \times T \to T$ and we assume that it is $\Gamma$-equivariant; it therefore yields a leafwise differential operator $D = (D_L)_{L \in V/\mathcal{F}}$ on $V$, which is elliptic along the leaves of $\mathcal{F}$. Notice that, if $\dim T > 0$ and $\Gamma = \{1\}$ then we are in the family situation; if $\dim T = 0$ and $\Gamma \neq \{1\}$, then we are in the covering situation; if $\dim T > 0$, $\Gamma \neq \{1\}$ and $T$ admits a $\Gamma$-invariant Borel measure $\nu$, then we are in the measured foliation situation and if $\dim T > 0$, $\Gamma \neq \{1\}$ then we are dealing with a more general foliation.

In the first three cases, there is first of all a *numeric* index: for families this is quite trivially the integral over $T$ of the locally constant function that associates to $\theta$ the index of $D_\theta$; for $\Gamma$-coverings we have the $\Gamma$-index of Atiyah and for measured foliations we have the measured index introduced by Connes. These last two examples involve the definition of a von Neumann algebra endowed with a suitable trace. More generally, and this applies also to general foliations, one can define *higher indices*, obtained by pairing the index class defined by an elliptic operator with suitable (higher) cyclic cocycles. In the case of foliated bundles there is a formula for these higher indices, due to Connes [**18**], and recently revisited by Gorokhovsky and Lott [**23**] using a generalization of the Bismut superconnection [**13**]. See also [**39**]. Since our main focus here are numeric (versus higher) invariants, we go back to the case of measured foliated bundles, thus assuming that $T$ admits a $\Gamma$-invariant measure $\nu$.

The index is of course a global object, defined in terms of the kernel and cokernel of operators. However, one of its essential features is the possibility of localizing it near the diagonal using the remainders produced by a parametrix for $D$. On a closed manifold this crucial property is encoded in the so-called Atiyah-Bott formula:

$$\operatorname{ind}(D) = \operatorname{Tr}(R_0^N) - \operatorname{Tr}(R_1^N)\,, \quad \forall N \geq 1 \tag{1}$$

if $R_1 = \mathrm{Id} - DQ$ and $R_0 = \mathrm{Id} - QD$ are the remainders of a parametrix $Q$. Similar results hold in the other two contexts: $\Gamma$-coverings and measured foliations. One important consequence of formula (1) and of the analogous one on $\Gamma$-coverings is Atiyah's index theorem on a $\Gamma$-covering $\tilde{M} \to M$, stating the equality of the index on $M$ and the von Neumann $\Gamma$-index on $\tilde{M}$. Informally, the index upstairs is equal to the index downstairs. On a measured foliation, for example on a foliated bundle $(\tilde{M} \times_\Gamma T, \mathcal{F})$ associated to a $\Gamma$-space $T$ endowed with a $\Gamma$-invariant measure $\nu$, we also have an index upstairs and an index downstairs, depending on whether we consider the $\Gamma$-equivariant family $(\tilde{D}_\theta)_{\theta\in T}$ or the longitudinal operator $D = (D_L)_{L\in V/\mathcal{F}}$; the analogue of formula (1) allows to prove the equality of these two indices. (This phenomenon is well known to experts; we explain it in detail in Section 4.)

Now, despite its many geometric applications, the index remains a very coarse spectral invariant of the elliptic differential operator $D$, depending only on the spectrum near zero. Especially when considering geometric operators, such as Dirac-type operators, and related geometric questions involving, for example, the diffeomorphism type of manifolds or the moduli space of metrics of positive scalar curvature, one is led to consider more involved spectral invariants. The eta invariant, introduced by Atiyah, Patodi and Singer on odd dimensional manifolds, is such an invariant. This invariant is highly non-local (in contrast to the index) and involves the whole spectrum of the operator. It is, however, too sophisticated: indeed, a small perturbation of the operator produces a variation of the corresponding eta invariant. In geometric questions one considers rather a more stable invariant, the rho invariant, typically a difference of eta invariants having the same local variation. The Cheeger-Gromov rho invariant on a Galois covering $\tilde{M} \to M$ of an odd dimensional manifold $M$ is the most famous example; it is precisely defined as the difference of the $\Gamma$-eta invariant on $\tilde{M}$, defined using the $\Gamma$-trace of Atiyah, and of the Atiyah-Patodi-Singer eta invariant of the base $M$. Notice that the analogous difference for the indices (in the even dimensional case) would be equal to zero because of Atiyah's index theorem on coverings; the Cheeger-Gromov rho invariant is thus a genuine *secondary invariant.* The Cheeger-Gromov rho invariant is usually defined for a Dirac-type operator $\tilde{D}$ and we bound ourselves to this case from now on; we denote it by $\rho_{(2)}(\tilde{D})$. Here are some of the stability properties of rho:

- let $(M, g)$ be an oriented riemannian manifold and let $\tilde{D}^{\rm sign}$ be the signature operator on $\tilde{M}$ associated to the $\Gamma$-invariant lift of $g$ to $\tilde{M}$: then $\rho_{(2)}(\tilde{D}^{\rm sign})$ is metric independent and a diffeomorphism invariant of $M$;
- let $M$ be a spin manifold and assume that the space $\mathcal{R}^+(M)$ of metrics with positive scalar curvature is non-empty. Let $g \in \mathcal{R}^+(M)$ and let $\tilde{D}_g^{\rm spin}$ be the spin Dirac operator associated to the $\Gamma$-invariant lift of $g$. Then the function $\mathcal{R}^+(M) \ni g \to \rho_{(2)}(\tilde{D}_g^{\rm spin})$ is constant on the connected components of $\mathcal{R}^+(M)$

There are easy examples, involving lens spaces, showing that $\rho_{(2)}(\tilde{D}^{\rm sign})$ is *not* a homotopy invariant and that $\mathcal{R}^+(M) \ni g \to \rho_{(2)}(\tilde{D}_g^{\rm spin})$ is *not* the constant function equal to zero. For purely geometric applications of these two results see, for example, **[15]** and **[46]**. These two properties can be proved in general, regardless of the nature of the group $\Gamma$. However, when $\Gamma$ is *torsion-free*, then the Cheeger-Gromov rho invariant enjoys particularly strong stability properties. Let $\Gamma = \pi_1(M)$ and let $\tilde{M} \to M$ be the universal cover. Then in a series of papers **[29]**, **[30]**, **[31]**, Keswani, extending work of Neumann **[41]**, Mathai **[36]** and Weinberger **[57]**, establishes the following fascinating theorem:

- if $M$ is orientable, $\Gamma$ is torsion free and the Baum-Connes map $K_*(B\Gamma) \to K_*(C^*_{\max}\Gamma)$ is an isomorphism, then $\rho_{(2)}(\tilde{D}^{\rm sign})$ is a *homotopy invariant* of $M$;
- if $M$ is in addition spin and $\mathcal{R}^+(M) \neq \varnothing$ then $\rho_{(2)}(\tilde{D}_g^{\rm spin}) = 0$ for any $g \in \mathcal{R}^+(M)$.

(The second statement is not explicitly given in the work of Keswani but it follows from what he proves; for a different proof of Keswani's result see the recent paper **[45]**.) Informally: *when $\Gamma$ is torsion free and the maximal Baum-Connes map is an isomorphism, the Cheeger-Gromov rho invariant behaves like an index, i.e. like a primary invariant: more precisely, it is a homotopy invariant for the signature operator and it is equal to zero for the spin Dirac operator associated to a metric of positive scalar curvature.*

Let us now move on in the hierarchy of geometric structures and consider a foliated bundle $(V := \tilde{M} \times_\Gamma T, \mathcal{F})$, with $\tilde{M} \to M$ the universal cover of an odd dimensional compact manifold and $T$ a compact $\Gamma$-space endowed with a $\Gamma$-invariant Borel (probability) measure $\nu$. We are also given a $\Gamma$-equivariant family of Dirac-type operators $\tilde{D} := (\tilde{D}_\theta)_{\theta \in T}$ on the product fibration $\tilde{M} \times T \to T$ and let $D = (D_L)_{L \in V/\mathcal{F}}$ be the induced longitudinally elliptic operator on $V$. One is then led to the following natural questions:

1. Can one define a foliated rho invariant $\rho_\nu(D; V, \mathcal{F})$?
2. What are its stability properties if $\tilde{D} = \tilde{D}^{\rm sign}$ and $\tilde{D} = \tilde{D}^{\rm spin}$?

3. If the isotropy groups of the action of $\Gamma$ on $T$ are torsion free and the maximal Baum-Connes map with coefficients
$$K_*^\Gamma(E\Gamma; C(T)) \to K_*(C(T) \rtimes_{\max} \Gamma)$$
is an isomorphism, is $\rho_\nu(V,\mathcal{F}) := \rho_\nu(D^{\mathrm{sign}}; V, \mathcal{F})$ a foliated homotopy invariant?

*The goal of this paper is to give an answer to these three questions. Along the way we shall present in a largely self-contained manner the main results in index theory and in the theory of eta invariants on foliated bundles.*

This work is organized as follows. In Section 1 we introduce the maximal $C^*$-algebra $\mathcal{A}_m$ associated to the $\Gamma$-space $T$ or, more precisely, to the groupoid $\mathcal{G} := T \rtimes \Gamma$. We endow this $C^*$-algebra with two traces $\tau^\nu_{\mathrm{reg}}$ and $\tau^\nu_{\mathrm{av}}$, $\nu$ denoting as before the $\Gamma$-invariant Borel measure on $T$. We then define two von Neumann algebras $W^*_{\mathrm{reg}}(\mathcal{G})$, $W^*_{\mathrm{av}}(\mathcal{G})$ with their respective traces; we define representations $\mathcal{A}_m \to W^*_{\mathrm{reg}}(\mathcal{G})$, $\mathcal{A}_m \to W^*_{\mathrm{av}}(\mathcal{G})$ and show compatibility of the traces involved.

In Section 2 we move to foliated bundles, giving the definition, studying the structure of the leaves, introducing the monodromy groupoid $G$ and the associated maximal $C^*$-algebra $\mathcal{B}_m$. We then introduce two von Neumann algebras, $W^*_\nu(G)$ and $W^*_\nu(V,\mathcal{F})$, to be thought of as the one upstairs and the one downstairs respectively, with their respective traces $\tau^\nu$, $\tau^\nu_{\mathcal{F}}$. We introduce representations $\mathcal{B}_m \to W^*_\nu(G)$, $\mathcal{B}_m \to W^*_\nu(V,\mathcal{F})$ and define two compatible traces, also denoted $\tau^\nu_{\mathrm{reg}}$ and $\tau^\nu_{\mathrm{av}}$, on the $C^*$-algebra $\mathcal{B}_m$. We then prove an explicit formula for these two traces on $\mathcal{B}_m$. We end Section 2 with a proof of the Morita isomorphism $K_0(\mathcal{A}_m) \simeq K_0(\mathcal{B}_m)$ and its compatibility with the morphisms
$$\tau^\nu_{\mathrm{reg},*}, \tau^\nu_{\mathrm{av},*} : K_0(\mathcal{A}_m) \to \mathbb{C}, \quad \tau^\nu_{\mathrm{reg},*}, \tau^\nu_{\mathrm{av},*} : K_0(\mathcal{B}_m) \to \mathbb{C}$$
induced by the two pairs of traces on $\mathcal{A}_m$ and $\mathcal{B}_m$ respectively.

In Section 3 we move to more analytic questions. We define a natural $\mathcal{A}_m$-Hilbert module $\mathcal{E}_m$ with associated $C^*$-algebra of compact operators $\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)$ isomorphic to $\mathcal{B}_m$; we show how $\mathcal{E}_m$ encodes both the $L^2$-spaces of the fibers of the product fibration $\tilde{M} \times T \to T$ and the $L^2$-spaces of the leaves of $\mathcal{F}$. We then introduce a $\Gamma$-equivariant pseudodifferential calculus, showing in particular how 0-th order operators extend to bounded $\mathcal{A}_m$-linear operators on $\mathcal{E}_m$ and how negative order operators extend to compact operators. We then move to *unbounded regular* operators, for example operators defined by a $\Gamma$-equivariant Dirac family $\tilde{D} := (\tilde{D}_\theta)_{\theta \in T}$ and study quite carefully the functional calculus associated to such an operator. We then treat Hilbert-Schmidt operators and trace class operators in our two von Neumann contexts and